Olympiad Inequalities

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It is the purpose of this document to familiarize the reader with a wide range of theorems and techniques that can be used to solve inequalities of the variety typically appearing on mathematical olympiads or other elementary proof contests. *The Standard Dozen* is an exhibition of twelve famous inequalities which can be cited and applied without proof in a solution. It is expected that most problems will fall entirely within the span of these inequalities. The *Examples* section provides numerous complete solutions as well as remarks on inequality-solving intuition, all intended to increase the reader's aptitude for the material covered here. It is organized in rough order of difficulty. Finally, the *Problems* section contains exercises without solutions, ranging from easy and straightforward to quite difficult, for the purpose of practicing techniques contained in this document.

I have compiled much of this from posts by my peers in a number of mathematical communities, particularly the Mathlinks-Art of Problem Solving forums,[1] as well as from various MOP lectures,[2] Kiran Kedlaya's inequalities packet,[3] and John Scholes' site.[4] I have tried to take note of original sources where possible. This work in progress is distributed for personal educational use only. In particular, any publication of all or part of this manuscript without explicit prior consent of the author, as well as any original sources noted herein, is strictly prohibited. Please send comments - suggestions, corrections, missing information, or other interesting problems - to the author at `tmildorfATmitDOTedu`.

Without further delay...

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[1] http://www.mathlinks.ro/Forum/ and http://www.artofproblemsolving.com respectively, though they have merged into a single, very large and robust group. The forums there are also host to a considerable wealth of additional material outside of inequalities.

[2] Math Olympiad Program. Although some people would try to convince me it is the Math Olympiad Summer Program and therefore is due the acronym MOSP, those who know acknowledge that the traditional "MOP" is the preferred appellation.

[3] The particularly diligent student of inequalities would be interested in this document, which is available online at http://www.unl.edu/amc/a-activities/a4-for-students/problemtext/ineqs-080299.tex. Further material is also available in the books Andreescu-Cirtoaje-Dospinescu-Lascu, *Old and New Inequalities*, GIL Publishing House, and Hardy-Littlewood-Pólya, *Inequalities*, Cambridge University Press. (The former is elementary and geared towards contests, the latter is more technical.)

[4] http://www.kalva.demon.co.uk/, where a seemingly inexhaustible supply of Olympiads is available.

1 The Standard Dozen

Throughout this lecture, we refer to *convex* and *concave* functions. Write I and I' for the intervals $[a, b]$ and (a, b) respectively. A function f is said to be convex on I if and only if $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function f that is continuous on I and twice differentiable on I' is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$ (Concave if the inequality is flipped.)

Let $x_1 \geq x_2 \geq \cdots \geq x_n$; $y_1 \geq y_2 \geq \cdots \geq y_n$ be two sequences of real numbers. If $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$ for $k = 1, 2, \ldots, n$ with equality where $k = n$, then the sequence $\{x_i\}$ is said to *majorize* the sequence $\{y_i\}$. An equivalent criterion is that for all real numbers t,

$$|t - x_1| + |t - x_2| + \cdots + |t - x_n| \geq |t - y_1| + |t - y_2| + \cdots + |t - y_n|$$

We use these definitions to introduce some famous inequalities.

Theorem 1 (Jensen) *Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, \ldots, x_n \in I$ and any nonnegative reals $\omega_1, \ldots, \omega_n$ with positive sum,*

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq (\omega_1 + \cdots + \omega_n) f\left(\frac{\omega_1 x_1 + \cdots + \omega_n x_n}{\omega_1 + \cdots + \omega_n}\right)$$

If f is concave, then the inequality is flipped.

Theorem 2 (Weighted Power Mean) *If $x_1, \ldots, x_n$ are nonnegative reals and $\omega_1, \ldots, \omega_n$ are nonnegative reals with a postive sum, then*

$$f(r) := \left(\frac{\omega_1 x_1^r + \cdots + \omega_n x_n^r}{\omega_1 + \cdots + \omega_n}\right)^{\frac{1}{r}}$$

is a non-decreasing function of r, with the convention that $r = 0$ is the weighted geometric mean. f is strictly increasing unless all the x_i are equal except possibly for $r \in (-\infty, 0]$, where if some x_i is zero f is identically 0. In particular, $f(1) \geq f(0) \geq f(-1)$ gives the AM-GM-HM inequality.

Theorem 3 (Hölder) *Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$; $\cdots$; $z_1, \ldots, z_n$ be sequences of nonnegative real numbers, and let $\lambda_a, \lambda_b, \ldots, \lambda_z$ positive reals which sum to 1. Then*

$$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_z} b_n^{\lambda_b} \cdots z_n^{\lambda_z}$$

This theorem is customarily identified as Cauchy when there are just two sequences.

Theorem 4 (Rearrangement) *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then, for any permutation π of $\{1, 2, \ldots, n\}$, we have*

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$$

with equality on the left and right holding if and only if the sequence $\pi(1), \ldots, \pi(n)$ is decreasing and increasing respectively.

Theorem 5 (Chebyshev) *Let $a_1 \le a_2 \le \cdots \le a_n$; $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers. Then*

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \ge \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}$$

Theorem 6 (Schur) *Let a, b, c be nonnegative reals and $r > 0$. Then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \ge 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

Remark - *This can be improved considerably. (See the problems section.) However, they are not as well known (as of now) as this form of Schur, and so should be proven whenever used on a contest.*

Theorem 7 (Newton) *Let $x_1, \ldots, x_n$ be nonnegative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x+x_1)(x+x_2)\cdots(x+x_n) = s_nx^n + \cdots + s_1x + s_0$, and define the symmetric averages by $d_i = s_i/\binom{n}{i}$. Then*

$$d_i^2 \ge d_{i+1}d_{i-1}$$

Theorem 8 (Maclaurin) *Let d_i be defined as above. Then*

$$d_1 \ge \sqrt{d_2} \ge \sqrt[3]{d_3} \ge \cdots \ge \sqrt[n]{d_n}$$

Theorem 9 (Majorization) *Let $f : I \to \mathbb{R}$ be a convex on I and suppose that the sequence $x_1, \ldots, x_n$ majorizes the sequence $y_1, \ldots, y_n$, where $x_i, y_i \in I$. Then*

$$f(x_1) + \cdots + f(x_n) \ge f(y_1) + \cdots + f(y_n)$$

Theorem 10 (Popoviciu) *Let $f : I \to \mathbb{R}$ be convex on I, and let $x, y, z \in I$. Then for any positive reals p, q, r,*

$$\begin{aligned} pf(x) + qf(y) + rf(z) \quad &+ \quad (p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right) \\ &\ge \quad (p+q)f\left(\frac{px+qy}{p+q}\right) + (q+r)f\left(\frac{qy+rz}{q+r}\right) + (r+p)f\left(\frac{rz+px}{r+p}\right) \end{aligned}$$

Theorem 11 (Bernoulli) *For all $r \ge 1$ and $x \ge -1$,*

$$(1+x)^r \ge 1 + xr$$

Theorem 12 (Muirhead) *Suppose the sequence $a_1, \ldots, a_n$ majorizes the sequence $b_1, \ldots, b_n$. Then for any positive reals $x_1, \ldots, x_n$,*

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

where the sums are taken over all permutations of n variables.

Remark *- Although Muirhead's theorem is a named theorem, it is generally not favorably regarded as part of a formal olympiad solution. Essentially, the majorization criterion guarantees that Muirhead's inequality can be deduced from a suitable application of AM-GM. Hence, whenever possible, you should use Muirhead's inequality only to deduce the correct relationship and then explicitly write all of the necessary applications of AM-GM. For a particular case this is a simple matter.*

We now present an array of problems and solutions based primarily on these inequalities and ideas.

2 Examples

When solving any kind of problem, we should always look for a comparatively easy solution first, and only later try medium or hard approaches. Although what constitutes this notoriously indeterminate "difficulty" varies widely from person to person, I usually consider "Dumbassing," AM-GM (Power Mean), Cauchy, Chebyshev (Rearrangement), Jensen, Hölder, in that order before moving to more clever techniques. (The first technique is described in remarks after example 1.) Weak inequalities will fall to AM-GM, which blatantly pins a sum to its smallest term. Weighted Jensen and Hölder are "smarter" in that the effect of widely unequal terms does not cost a large degree of sharpness[5] - observe what happens when a weight of 0 appears. Especially sharp inequalities may be assailable only through clever algebra.

Anyway, I have arranged the following with that in mind.

1. Show that for positive reals a, b, c

$$\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right) \geq 9a^2b^2c^2$$

Solution 1. Simply use AM-GM on the terms within each factor, obtaining

$$\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right) \geq \left(3\sqrt[3]{a^3b^3c^3}\right)\left(3\sqrt[3]{a^3b^3c^3}\right) = 9a^2b^2c^2$$

[5]The sharpness of an inequality generally refers to the extent to which the two sides mimic each other, particularly near equality cases.

Solution 2. Rearrange the terms of each factor and apply Cauchy,

$$\left(a^2b+b^2c+c^2a\right)\left(bc^2+ca^2+ab^2\right) \geq \left(\sqrt{a^3b^3c^3}+\sqrt{a^3b^3c^3}+\sqrt{a^3b^3c^3}\right)^2 = 9a^2b^2c^2$$

Solution 3. Expand the left hand side, then apply AM-GM, obtaining

$$\begin{aligned}\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) &= a^3b^3+a^2b^2c^2+a^4bc \\ &+ ab^4c+b^3c^3+a^2b^2c^2 \\ &+ a^2b^2c^2+abc^4+a^3c^3 \\ &\geq 9\sqrt[9]{a^{18}b^{18}c^{18}} = 9a^2b^2c^2\end{aligned}$$

We knew this solution existed by Muirhead, since $(4,1,1),(3,3,0)$, and $(2,2,2)$ all majorize $(2,2,2)$. The strategy of multiplying out all polynomial expressions and applying AM-GM in conjunction with Schur is generally knowing as dumbassing because it requires only the calculational fortitude to compute polynomial products and no real ingenuity. As we shall see, dumbassing is a valuable technique. We also remark that the AM-GM combining all of the terms together was a particularly weak inequality, but the desired was a multiple of $a^2b^2c^2$'s, the smallest 6th degree symmetric polynomial of three variables; such a singular AM-GM may not always suffice.

2. Let a,b,c be positive reals such that $abc=1$. Prove that

$$a+b+c \leq a^2+b^2+c^2$$

Solution. First, we *homogenize* the inequality. that is, apply the constraint so as to make all terms of the same degree. Once an inequality is homogenous in degree d, we may scale all of the variables by an arbitrary factor of k, causing both sides of the inequality to scale by the factor k^d. This is valid in that it does not change the correctness of an inequality for any positive k, and if d is even, for any nonzero k. Hence, we need consider a nonhomogenous constraint no futher. In this case, we multiply the left hand side by $\sqrt[3]{abc}$, obtaining

$$a^{\frac{4}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{4}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{4}{3}} \leq a^2+b^2+c^2$$

As $abc=1$ is not homogenous, the above inequality must be true for all nonnegative a,b,c. As $(2,0,0)$ majorizes $(4/3,1/3,1/3)$, we know it is true, and the necessary AM-GM is

$$\frac{2a^2}{3}+\frac{b^2}{6}+\frac{c^2}{6} = \frac{a^2+a^2+a^2+a^2+b^2+c^2}{6} \geq \sqrt[6]{a^8b^2c^2} = a^{\frac{4}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

3. Let $P(x)$ be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$.

Solution. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The first thing we notice is that the given is $P(1) \geq 1$. Hence, the natural strategy is to combine $P(x)$ and $P\left(\frac{1}{x}\right)$ into $P(1)$ in some fashion. The best way to accomplish this is Cauchy, which gives

$$\begin{aligned} P(x)P\left(\frac{1}{x}\right) &= (a_n x^n + \cdots + a_1 x + a_0)\left(a_n \frac{1}{x^n} + \cdots + a_1 \frac{1}{x} + a_0\right) \\ &\geq (a_n + \cdots + a_1 + a_0)^2 = P(1)^2 \geq 1 \end{aligned}$$

as desired. This illustrates a useful means of eliminating denominators - by introducing similar factors weighted by reciprocals and applying Cauchy / Hölder.

4. (USAMO 78/1) a, b, c, d, e are real numbers such that

$$\begin{aligned} a + b + c + d + e &= 8 \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 16 \end{aligned}$$

What is the largest possible value of e?

Solution. Observe that the givens can be effectively combined by considering squares:

$$\begin{aligned} (a-r)^2 + (b-r)^2 + (c-r)^2 + (d-r)^2 + (e-r)^2 &= (a^2 + b^2 + c^2 + d^2 + e^2) \\ &- 2r(a + b + c + d + e) + 5r^2 \\ &= 16 - 16r + 5r^2 \end{aligned}$$

Since these squares are nonnegative, $e \leq \sqrt{5r^2 - 16r + 16} + r = f(r)$ for all r. Since equality $e = f(r)$ can be achieved when $a = b = c = d = r$, we need only compute the smallest value $f(r)$. Since f grows large at either infinity, the minimum occurs when $f'(r) = 1 + \frac{10r-16}{2\sqrt{5r^2-16r+16}} = 0$. The resultant quadratic is easily solved for $r = \frac{6}{5}$ and $r = 2$, with the latter being an extraneous root introduced by squaring. The largest possible e and greatest lower bound of $f(r)$ is then $f(6/5) = 16/5$, which occurs when $a = b = c = d = 6/5$ and $e = 16/5$. Alternatively, proceed as before except write $a = b = c = d = \frac{8-e}{4}$ since the maximum e must occur when the other four variables are equal. The second condition becomes a quadratic, and the largest solution is seen to be $e = \frac{16}{5}$.

The notion of equating a, b, c, d is closely related to the idea of *smoothing* and Jensen's inequality. If we are working with $S_1 = f(x_1) + \cdots + f(x_n)$ under the constraint of a fixed sum $x_1 + \cdots + x_n$, we can decrease S_1 by moving several x_i in the same interval I together (that is, replacing $x_{i_1} < x_{i_2}$ with $x'_{i_1} = x_{i_1} + \epsilon < x_{i_2} - \epsilon = x'_{i_2}$ for any sufficiently small ϵ) for any I where f is convex. S_1 can also be decreased by spreading x_i in the same interval where f is concave. When seeking the maximum of S_1, we proceed in the opposite fashion, pushing x_i on the concave intervals of f together and moving x_i on the convex intervals apart.

5. Show that for all positive reals a, b, c, d,

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$$

Solution. Upon noticing that the numerators are all squares with $\sqrt{1} + \sqrt{1} + \sqrt{4} + \sqrt{16} = \sqrt{64}$, Cauchy should seem a natural choice. Indeed, multiplying through by $a + b + c + d$ and applying Cauchy, we have

$$(a+b+c+d)\left(\frac{1^2}{a} + \frac{1^2}{b} + \frac{2^2}{c} + \frac{4^2}{d}\right) \geq (1+1+2+4)^2 = 64$$

as desired.

6. (USAMO 80/5) Show that for all non-negative reals $a, b, c \leq 1$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1$$

Solution. Let $f(a, b, c)$ denote the left hand side of the inequality. Since $\frac{\partial^2}{\partial a^2} f = \frac{2b}{(c+a+1)^3} + \frac{2c}{(a+b+1)^3} \geq 0$, we have that f is convex in each of the three variables; hence, the maximum must occur where $a, b, c \in \{0, 1\}$. Since f is 1 at each of these 8 points, the inequality follows.

Second derivative testing for convexity/concavity is one of the few places where the use of Calculus is not seriously loathed by olympiad graders. It is one of the standard techniques in inequalities and deserves to be in any mental checklist of inequality solving. In this instance, it led to an easy solution.

7. (USAMO 77/5) If a, b, c, d, e are positive reals bounded by p and q with $0 < p \leq q$, prove that

$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$

and determine when equality holds.

Solution. As a function f of five variables, the left hand side is convex in each of a, b, c, d, e; hence, its maximum must occur when $a, b, c, d, e \in \{p, q\}$. When all five variables are p or all five are q, f is 25. If one is p and the other four are q, or vice versa, f becomes $17 + 4(\frac{p}{q} + \frac{q}{p})$, and when three are of one value and two of the other, $f = 13 + 6(\frac{p}{q} + \frac{q}{p})$. $\frac{p}{q} + \frac{q}{p} \geq 2$, with equality if and only if $p = q$. Clearly, equality holds where $p = q$. Otherwise, the largest value assumed by f is $13 + 6(\frac{p}{q} + \frac{q}{p})$, which is obtained only when two of a, b, c, d, e are p and the other three are q, or vice versa. In such instances, f is identically the right hand side.

This is a particular case of the Schweitzer inequality, which, in its weighted form, is sometimes known as the Kantorovich inequality. Again, the proof is hardly difficult.

8. $a, b, c,$ are non-negative reals such that $a + b + c = 1$. Prove that

$$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}$$

Solution. Multiplying by 4 and homogenizing, we seek

$$\begin{aligned} 4a^3 + 4b^3 + 4c^3 + 24abc &\geq (a+b+c)^3 \\ &= a^3 + b^3 + c^3 + 3\left(a^2(b+c) + b^2(c+a) + c^2(a+b)\right) + 6abc \\ \iff a^3 + b^3 + c^3 + 6abc &\geq a^2(b+c) + b^2(c+a) + c^2(a+b) \end{aligned}$$

Recalling that Schur's inequality gives $a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)$, the inequality follows. In particular, equality necessitates that the extra $3abc$ on the left is 0. Combined with the equality condition of Schur, we have equality where two of a, b, c are $\frac{1}{2}$ and the third is 0. This is a typical dumbass solution.

Solution 2. Without loss of generality, take $a \geq b \geq c$. As $a + b + c = 1$, we have $c \leq \frac{1}{3}$ or $1 - 3c \geq 0$. Write the left hand side as $(a + b)^3 - 3ab(a + b - 2c) = (a+b)^3 - 3ab(1-3c)$. This is minimized for a fixed sum $a+b$ where ab is made as large as possible. As by AM-GM $(a + b)^2 \geq 4ab$, this minimum occurs if and only if $a = b$. Hence, we need only consider the one variable inequality $2\left(\frac{1-c}{2}\right)^3 + c^3 + 6\left(\frac{1-c}{2}\right)^2 c = \frac{1}{4} \cdot (9c^3 - 9c^2 + 3c + 1) \geq \frac{1}{4}$. Since $c \leq \frac{1}{3}$, $3c \geq 9c^2$. Dropping this term and $9c^3$, the inequality follows. Particularly, $9c^3 = 0$ if and only if $c = 0$, and the equality cases are when two variables are $\frac{1}{2}$ and the third is 0. This solution is of the smoothing variety in that we moved two variables together while preserving their sum. In other inequalities we may wish to preserve products and thus analyze the assignment $a' = b' = \sqrt{ab}$. These are both representative of the technique of mixing variables, in which two or more variables are blended together algebraically as part of an argument allowing us to outright equate variables.

9. (IMO 74/5) If a, b, c, d are positive reals, then determine the possible values of

$$\frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

Solution. We can obtain any real value in $(1, 2)$. The lower bound is approached by $a \to \infty$, $b = d = \sqrt{a}$, and $c = 1$. The upper bound is approached by $a = c \to \infty$, $b = d = 1$. As the expression is a continuous function of the variables, we can obtain all of the values in between these bounds. Finally, these bounds are strict because

$$\begin{aligned} &\frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d} > \\ &\frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1 \end{aligned}$$

and

$$\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}<$$
$$\frac{a}{a+b}+\frac{b}{a+b}+\frac{c}{c+d}+\frac{d}{c+d}=2$$

Whenever extrema occur for unusual parameterizations, we should expect the need for non-classical inequalities such as those of this problem where terms were completely dropped.

10. (IMO 95/2) a, b, c are positive reals with $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

Solution 1. Let $x = \frac{1}{a}, y = \frac{1}{b}$, and $z = \frac{1}{c}$. We perform this substitution to move terms out of the denominator. Since $abc = xyz = 1$, we have

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}=\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}$$

Now, multiplying through by $(x + y)(y + z)(z + x)$, we seek

$$x^4+y^4+z^4+x^3y+x^3z+y^3z+xy^3+xz^3+yz^3+x^2yz+xy^2z+xyz^2\geq$$
$$\sqrt[3]{xyz}\cdot\left(3xyz+\frac{3}{2}\cdot\left(x^2y+x^2z+y^2x+xy^2+xz^2+yz^2\right)\right)$$

which follows immediately by AM-GM, since $x^2yz+xy^2z+xyz^2 \geq 3\sqrt[3]{x^4y^4z^4}$, $\frac{x^3y+xy^3+x^3z}{3} \geq \sqrt[3]{x^7y^4z}$ and $\frac{7x^4+4y^4+z^4}{12} \geq \sqrt[3]{x^7y^4z}$ - as guaranteed by Muirhead's inequality.

Solution 2. Substitute x, y, z as before. Now, consider the convex function $f(x) = x^{-1}$ for $x > 0$. ($f(x) = x^c$ is convex for $c < 0$ and $c \geq 1$, and concave for $0 < c \leq 1$, verify this with the second derivative test.) Now, by Jensen,

$$\begin{aligned}\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} &= xf\left(\frac{y+z}{x}\right)+yf\left(\frac{z+x}{y}\right)+zf\left(\frac{x+y}{z}\right)\\ &\geq (x+y+z)f\left(\frac{(y+z)+(z+x)+(x+y)}{x+y+z}\right)=\frac{x+y+z}{2}\end{aligned}$$

But $x + y + z \geq 3\sqrt[3]{xyz} = 3$, as desired.

Solution 3. Perform the same substitution. Now, multiplying by $(x + y + z)$ and applying Cauchy, we have

$$\frac{1}{2}\left((y+z)+(z+x)+(x+y)\right)\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\ \geq\ \frac{1}{2}(x+y+z)^2$$

Upon recalling that $x+y+z \geq 3$ we are done. Incidentally, the progress of this solution with Cauchy is very similar to the weighted Jensen solution shown above. This is no coincidence, it happens for many convex $f(x) = x^c$.

Solution 4. Apply the same substitution, and put $x \geq y \geq z$. Simultaneously, $\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}$. Hence, by Chebyshev,

$$x \cdot \left(\frac{x}{y+z}\right) + y \cdot \left(\frac{y}{z+x}\right) + z \cdot \left(\frac{z}{x+y}\right) \geq \frac{x+y+z}{3}\left(\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}\right)$$

Again, $x+y+z \geq 3$. But now we have Nesbitt's inequality, $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$. This follows immediately from AM-HM upon adding 1 to each term. Suffice it to say that with such a multitude of solutions, this problem was hardly difficult.

11. Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{2}{(a+1)^2 + b^2 + 1} + \frac{2}{(b+1)^2 + c^2 + 1} + \frac{2}{(c+1)^2 + a^2 + 1} \leq 1$$

Solution. The previous problem showed the substitution offers a way to rewrite an inequality in a more convenient form. Substitution can also be used to implicity use a given. First, expand the denominators and apply AM-GM, obtaining

$$\frac{2}{(a+1)^2 + b^2 + 1} = \frac{2}{a^2 + b^2 + 2a + 2} \leq \frac{1}{ab + a + 1}$$

Now, write $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$. We have $\frac{1}{ab+a+1} = \frac{1}{\frac{x}{z}+\frac{x}{y}+1} = \frac{yz}{xy+yz+zx}$. It is now evident that the sum of the new fractions is 1.

That positive reals a, b, c have product 1 if and only if $\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} = 1$ is a curious and recurring identity. Also to be taken from this problem is the idea of "isolated fudging."[6] In many inequalities we are content to use Cauchy or Jensen, which blend n terms in a sum together, but in others it is prudent to restrict our attention to individual terms.

12. (USAMO 98/3) Let $a_0, \ldots, a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1$$

Prove that

$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}$$

[6] This term coined at MOP 2005 by Hyun Soo Kim.

Solution 1. Let $y_i = \tan\left(x - \frac{\pi}{4}\right)$. We have $\tan(x_i) = \tan\left((x_i - \frac{\pi}{4}) + \frac{\pi}{4}\right) = \frac{y_i+1}{1-y_i}$. Hence, given $s = y_0 + \cdots + y_n \geq n - 1$ we seek to prove $\prod_{i=0}^{n} \frac{1+y_i}{1-y_i} \geq n^{n+1}$. Observe that for any $a > b$ under a fixed sum $a + b$, the expression

$$\left(\frac{1+a}{1-a}\right) \cdot \left(\frac{1+b}{1-b}\right) = 1 + \frac{2(a+b)}{(1-a)(1-b)}$$

can be decreased by moving a and b any amount closer together. Hence, for any sequence $y_0, \ldots, y_n$, we can replace any $y_i > \frac{s}{n+1}$ and $y_j < \frac{s}{n+1}$ with $y_i' = \frac{s}{n+1}$ and $y_j' = y_i + y_j - \frac{s}{n+1}$, decreasing the product. Since n is finite this process terminates where all of the variables are equal. Thus, we have

$$\begin{aligned} \prod_{i=0}^{n} \frac{1+y_i}{1-y_i} &\geq \left(\frac{1+\frac{s}{n+1}}{1-\frac{s}{n+1}}\right)^{n+1} \\ &\geq \left(\frac{\frac{2n}{n+1}}{\frac{2}{n+1}}\right)^{n+1} = n^{n+1} \end{aligned}$$

Where the last inequality follows from the fact that $\frac{1+x}{1-x}$ is an increasing function of x.

Solution 2. Perform the same substitution. The given can be written as $1 + y_i \geq \sum_{j \neq i}(1 - y_j)$, which by AM-GM gives $\frac{1+y_n}{n} \geq \prod_{j \neq i} (1 - y_j)^{\frac{1}{n}}$. (There are $n + 1$ variables.) Now we have

$$\prod_{i=0}^{n} \frac{1+y_i}{n} \geq \prod_{i=0}^{n} \prod_{j \neq i} (1 - y_j)^{\frac{1}{n}} = \prod_{i=0}^{n} (1 - y_i)$$

as desired. Our first solution again used the idea of smoothing and was fairly natural, whereas the second is a bit magical.

13. Let a, b, c be positive reals. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

with equality if and only if $a = b = c = 1$.

Solution. Multiply through by $1+abc$ and add three to each side, on the left obtaining

$$\begin{aligned} & \frac{1+a+ab+abc}{a(1+b)} + \frac{1+b+bc+abc}{b(1+c)} + \frac{1+c+ac+abc}{c(1+a)} \\ = & \frac{(1+a)+ab(1+c)}{a(1+b)} + \frac{(1+b)+bc(1+a)}{b(1+c)} + \frac{(1+c)+ac(1+b)}{c(1+a)} \end{aligned}$$

which is at least 6 by AM-GM, as desired. In particular, this AM-GM asserts the equivalence of $\frac{(1+a)}{a(1+b)}$ and $\frac{a(1+b)}{1+a}$, or that they are both one. Then $a + ab = 1 + a$ so

$ab = 1$. It follows easily that $a = b = c = 1$ is the unique equality case. Here, we used algebra to connect the 1 and abc by adding a term of first degree and a term of second degree to each numerator. Dumbassing is made nontrivial due to the asymmetry involved, so we put clearing all denominators on hold. Multiplying by only the $1 + abc$ term was reasonably well-motivated, however, because it is unclear how one will blend the denominators on the left to obtain this factor.

14. (Romanian TST) Let a, b, x, y, z be positive reals. Show that

$$\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \geq \frac{3}{a + b}$$

Solution. Note that $(a + b)(xy + yz + xz) = (x(ay + bz) + y(az + bx) + z(ax + by))$. We introduce this factor in the inequality, obtaining

$$(x(ay + bz) + y(az + bx) + z(ax + by)) \left(\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \right) \geq$$
$$(x + y + z)^2 \geq 3(xy + yz + xz)$$

Where the last inequality is simple AM-GM. The desired follows by simple algebra. Again we have used the idea of introducing a convenient factor to clear denominators with Cauchy.

15. The numbers $x_1, x_2, \ldots, x_n$ obey $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3}$$

Solution 1. Substitute $y_i = x_i^3$ so that $y_1 + \cdots + y_n = 0$. In maximizing $\sqrt[3]{y_1} + \cdots + \sqrt[3]{y_n}$, we note that $f(y) = y^{\frac{1}{3}}$ is concave over $[0, 1]$ and convex over $[-1, 0]$, with $|f'(y_1)| \geq |f'(y_2)| \iff 0 < |y_1| \leq |y_2|$. Hence, we may put $y_1 = \cdots = y_k = -1$; $-1 \leq y_{k+1} < 0$, and $y_{k+2} = \cdots = y_n = \frac{k - y_{k+1}}{n-k-1}$. We first show that y_{k+1} leads to a maximal sum of $\sqrt[3]{y_i}$ if it is -1 or can be made positive. If $|y_{k+1}| < |y_{k+2}|$, we set $y'_{k+1} = y'_{k+2} = \frac{y_{k+1} + y_{k+2}}{2}$, increasing the sum while making y_{k+1} positive. Otherwise, set $y'_{k+1} = -1$ and $y'_{k+2} = 1 - y_{k+1} - y_{k+2}$, again increasing the sum of the $\sqrt[3]{y_i}$. Now we may apply Jensen to equate all positive variables, so that we need only show

$$k\sqrt[3]{-1} + (n - k)\sqrt[3]{\frac{k}{n - k}} \leq \frac{n}{3}$$

But we have

$$(n + 3k)^3 - 27(n - k)^2 k = n^3 - 18n^2 k + 81nk^2 = n(n - 9k)^2 \geq 0$$

as desired. Particularly, as k is an integer, equality can hold only if $9|n$ and then if and only if one ninth of the variables y_i are -1 and the rest are 1/8.

Solution 2. Let $x_i = \sin(\alpha_i)$, and write $0 = x_1^3 + \cdots + x_n^3 = \sin^3(\alpha_1) + \cdots + \sin^3(\alpha_n) = \frac{1}{4}\left((3\sin(\alpha_1) - \sin(3\alpha_1)) + \cdots + (3\sin(\alpha_n) - \sin(3\alpha_n))\right)$. It follows that $x_1 + \cdots + x_n = \sin(\alpha_1) + \cdots + \sin(\alpha_n) = \frac{\sin(3\alpha_1) + \cdots + \sin(3\alpha_n)}{3} \le \frac{n}{3}$. The only values of $\sin(\alpha)$ which lead to $\sin(3\alpha) = 1$ are $\frac{1}{2}$ and -1. The condition for equality follows.

The first solution is a fairly straightforward, if ugly, solution based on the ideas of smoothing and mixing variables. The reader is encouraged to grow familiar with such methods as nearly all functions appearing in Olympiad inequalities involve a limited number of inflexion points and thus open themselves up to such lines of attack. Although the latter is cleaner, more elegant, and perhaps worthy of more praise, it provides little use beyond this problem.

16. (Turkey) Let $n \ge 2$ be an integer, and $x_1, x_2, \ldots, x_n$ positive reals such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Determine the smallest possible value of

$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_3 + \cdots + x_n + x_1} + \cdots + \frac{x_n^5}{x_1 + \cdots + x_{n-1}}$$

Solution. Observe that $\sum_{i=1}^n x_i \sum_{j \ne i} x_j \le n - 1$, so that

$$\begin{aligned}
\left(\sum_{i=1}^n x_i \left(\sum_{j\ne i} x_j\right)\right)\left(\sum_{i=1}^n \frac{x_i^5}{\sum_{j\ne i} x_i}\right) &\ge \left(x_1^3 + \cdots + x_n^3\right)^2 \\
&= n^2 \left(\frac{x_1^3 + \cdots + x_n^3}{n}\right)^2 \\
&\ge n^2 \left(\frac{x_1^2 + \cdots + x_n^2}{n}\right)^3 = \frac{1}{n}
\end{aligned}$$

Leads to

$$\sum_{i=1}^n \frac{x_i^5}{\sum_{j\ne i} x_i} \ge \frac{1}{n(n-1)}$$

with equality if and only if $x_1 = \cdots = x_n = \frac{1}{\sqrt{n}}$. If the reader is bored by repeatedly seeing the same Cauchy-clears-denominators punchline, then perhaps it is because the author has succeeded in so inculcating its potency that the reader will never forget it! The reasons for weighting each denominator with the first power of the corresponding x_i are twofold. On the convenience side, it leads to an integer exponent later, and on the necessity side, it allows us to bound our new factor above because it has the same degree as our given. Last, the application of Power-mean is the step required to eliminate all variables by introducing our inhomogenous given equation.

17. (IMO Shortlist) Find the minimum value of c such that for any n and any nonnegative reals $x_1, x_2, \ldots, x_n$ which satisfy $x_{i+1} \ge x_1 + x_2 + \cdots + x_i$ for $i = 1, \ldots, n-1$, we have

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \le c\sqrt{x_1 + x_2 + \cdots + x_n}$$

Solution. We claim that $c = 1+\sqrt{2}$. First, we show that this is a lower bound by considering $x_1 = 1$, $x_k = 2^{k-2}$ for $k \geq 2$. We have

$$\sum_{i=1}^{n} \sqrt{x_i} = 1 + \frac{2^{\frac{n-1}{2}} - 1}{\sqrt{2}-1} = 2^{\frac{n-1}{2}}(\sqrt{2}+1) - \sqrt{2} \;\leq\; c\sqrt{x_1 + \cdots + x_n} = c2^{\frac{n-1}{2}}$$

$$\frac{2^{\frac{n-1}{2}}(1+\sqrt{2}) - \sqrt{2}}{2^{\frac{n-1}{2}}} \leq c \qquad (*)$$

$(*)$ must hold for all positive integers n, so $c \geq 1+\sqrt{2}$. We prove by induction that this value of c is sufficient. The base case $n = 1$ is trivial, so we need only prove the claim for $n = n_0 + 1$ given that it holds for $n = n_0$. By hypothesis, we have

$$s\sqrt{x_1} + \cdots + \sqrt{x_{n_0}} \leq (1+\sqrt{2})\sqrt{x_1 + \cdots + x_{n_0}}$$

Let $\alpha = x_1 + \cdots + x_{n_0}$ and $\beta = x_{n_0+1}$, where $\beta \geq \alpha$. Now

$$\begin{aligned}
(1+\sqrt{2})\sqrt{\alpha} + \sqrt{\beta} \;&\leq\; (1+\sqrt{2})\sqrt{\alpha+\beta} \qquad (**) \\
\iff \sqrt{\beta} \;&\leq\; (1+\sqrt{2})\left(\sqrt{\alpha+\beta} - \sqrt{\alpha}\right) \\
\iff \frac{\sqrt{\beta}}{\sqrt{\alpha+\beta} - \sqrt{\alpha}} \;&\leq\; 1+\sqrt{2} \\
\iff \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\alpha}{\beta}+1} \;&\leq\; 1+\sqrt{2} \qquad (***)
\end{aligned}$$

The left hand side of $(***)$ is a decreasing function of β, and thus assumes its largest possible value when β is made as small as possible. But because $\beta \geq \alpha$, this extremum is $1+\sqrt{2}$. This proves that $(**)$ is valid as well, completing our induction.

This problem serves as a reminder that inequalities can be proven by induction. Indeed, it seems almost a working theorem that problems containing an italicized n can be solved by induction, or the equivalent.

18. (Poland 95) Let n be a positive integer. Compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}$$

where $x_1, x_2, \ldots, x_n$ are positive reals such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n$$

Solution. The given is that the harmonic mean of $x_1, \ldots, x_n$ is 1, which implies that the product $x_1 x_2 \cdots x_n$ is at least 1. Now, we apply weighted AM-GM

$$\begin{aligned}
x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n} \;&\geq\; \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \sqrt[1+\frac{1}{2}+\cdots+\frac{1}{n}]{x_1 x_2 \cdots x_n} \\
&\geq\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\end{aligned}$$

The solution is a natural way of using the likeness of the exponents and denominators, especially after we have interpreted the hideous given in a manner more to our liking. We should not be afraid to lose the precision so long as we are confident that the extremal value does occur at $a_1 = a_2 = \cdots = a_n = 1$.

19. Prove that for all positive reals a, b, c, d,

$$a^4b + b^4c + c^4d + d^4a \geq abcd(a + b + c + d)$$

Solution. By AM-GM,

$$\frac{23a^4b + 7b^4c + 11c^4d + 10ad^4}{51} \geq \sqrt[51]{a^{102}b^{51}c^{51}d^{51}} = a^2bcd$$

from which the desired follows easily. Indeed, the most difficult part of this problem is determining suitable weights for the AM-GM. One way is to suppose arbitrary weights x_1, x_2, x_3, x_4 for a^4b, b^4c, c^4d, ad^4 respectively, and solve the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 4x_1 + x_2 &= 2 \\ 4x_2 + x_3 &= 1 \\ 4x_3 + x_4 &= 1 \end{aligned}$$

20. (USAMO 01/3) Let a, b, c be nonnegative reals such that

$$a^2 + b^2 + c^2 + abc = 4$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2$$

Solution [by Tony Zhang.] For the left hand side, note that we cannot have $a, b, c > 1$. Suppose WLOG that $c \leq 1$. Then $ab+bc+ca-abc \geq ab+bc+ca-ab = c(a+b) \geq 0$. For the right, $4 = a^2 + b^2 + c^2 + abc \geq 4(abc)^{\frac{3}{4}} \implies abc \leq 1$. Since by the pigeon hole principle, among three numbers either two exceed 1 or two are at most 1. Hence, we assume WLOG that $(a-1)(b-1) \geq 0$, which gives $ab+1 \geq a+b \iff abc+c \geq ac+bc \iff c \geq ac+bc-abc$. Now, we have $ab+bc+ca-abc \leq ab+c$. Either we are done or $ab+c > 2$. But in the latter case, $4 = (a^2+b^2)+c(c+2ab) > 2ab+2c = 2(ab+c) > 4$, a contradiction.

The lower bound was easy, but the upper bound required considerable asymmetric manipulation. Such work may be necessary in the presence of complicated and asymmetric givens. Although the above solution is not without traces of magic, it may be possible to substitute the sheer cleverness of arguments such as $(a-1)(b-1) \geq 0$ with several cases.

21. (Vietnam 98) Let $x_1, \ldots, x_n$ be positive reals such that

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+\cdots+\frac{1}{x_n+1998}=\frac{1}{1998}$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998$$

Solution. Let $y_i = \frac{1}{x_i+1998}$ so that $y_1 + \cdots + y_n = \frac{1}{1998}$ and $x_i = \frac{1}{y_i} - 1998$. Now

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n}\left(\frac{1}{y_i} - 1998\right) = e^{\sum_{i=1}^{n}\ln\left(\frac{1}{y_i}-1998\right)}$$

Hence, to minimize the product of the x_i, we equivalently minimize the sum of $\ln\left(\frac{1}{y_i} - 1998\right)$. In particular,

$$\begin{aligned}
\frac{d}{dy}\left(\ln\left(\frac{1}{y}-1998\right)\right) &= \frac{1}{\left(\frac{1}{y}-1998\right)^2}\cdot\frac{-1}{y^2} \\
&= \frac{-1}{y-1998y^2} \\
\frac{d^2}{dy^2}\left(\ln\left(\frac{1}{y}-1998\right)\right) &= \frac{1-3996y}{(y-1998y^2)^2}
\end{aligned}$$

So $\ln\left(\frac{1}{y}-1998\right)$ is convex on $[0, 1/3996]$. If we had $0 < y_i \leq 1/3996$ for all i we could apply Jensen. Since $y_i + y_j \leq 1/1998$ for all i, j, we consider

$$\begin{aligned}
\left(\frac{1}{a}-1998\right)\left(\frac{1}{b}-1998\right) &\geq \left(\frac{2}{a+b}-1998\right)^2 \\
\iff \frac{1}{ab} - 1998\left(\frac{1}{a}+\frac{1}{b}\right) &\geq \frac{4}{(a+b)^2} - \frac{4\cdot 1998}{a+b} \\
\iff (a+b)^2 - 1998(a+b)^3 &\geq 4ab - 4ab(a+b)\cdot 1998 \\
\iff (a-b)^2 \geq 1998(a+b)(a-b)^2 &
\end{aligned}$$

which incidentally holds for any $a+b \leq \frac{1}{1998}$. Hence, any two y_i and y_j may be set to their average while decreasing the sum in question; hence, we may assume $y_i \in (0, \frac{1}{3996}]$. Now Jensen's inequality shows that the minimum occurs when $y_i = \frac{1}{1998n}$ for all i, or when $x_i = 1998(n-1)$ for all i. It is easy to see that this yields equality.

In this problem, we used mixing variables to get around the difficulties presented by the inflexion point. This is a useful idea to enhance our capabilities in applying smoothing and Jensen.

22. (Romania 99) Show that for all positive reals $x_1, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$, we have

$$\frac{1}{n-1+x_1} + \cdots + \frac{1}{n-1+x_n} \leq 1$$

Solution. First, we prove a lemma: the maximum of the sum occurs when $n-1$ of the x_i are equal. Consider $f(y) = \frac{1}{k+e^y}$ for an arbitrary nonnegative constant k. We have $f'(y) = \frac{-e^y}{(k+e^y)^2}$ and $f''(y) = \frac{e^y(e^y-k)}{(k+e^y)^3}$. Evidently $f''(y) \geq 0 \iff e^y \geq k$. Hence, $f(y)$ has a single inflexion point where $y = \ln(k)$, where $f(y)$ is convex over the interval $((\ln(k), \infty)$. Now, we employ the substitution $y_i = \ln(x_i)$ so that $y_1 + \cdots + y_n = 0$ and

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} = \sum_{i=1}^{n} f(y_i)$$

We take $k = n-1$ and write $k_0 = \ln(n-1)$. Suppose that $y_1 \geq \cdots \geq y_m \geq k_0 \geq y_{m+1} \geq \cdots x_n$ for some positive m. Then by, Majorization,

$$f(y_1) + \cdots + f(y_m) \leq (m-1)f(k_0) + f(y_1 + \cdots + y_m - (m-1)k_0)$$

But then, also by Majorization,

$$(m-1)f(k_0) + f(y_{m+1}) + \cdots + f(y_n) \leq (n-1)f\left(\frac{(m-1)k_0 + y_{m+1} + \cdots + y_n}{n-1}\right)$$

Otherwise, all of the y_i are less than k_0. In that case we may directly apply Majorization to equate $n-1$ of the y_i whilst increasing the sum in question. Hence, the lemma is valid.[7] ▲

Applying the lemma, it would suffice to show

$$\frac{k}{k+x} + \frac{1}{k + \frac{1}{x^k}} \leq 1$$

Clearing the denominators,

$$\begin{aligned}\left(k^2 + \frac{k}{x^k}\right) + (k+x) \quad &\leq \quad k^2 + k\left(x + \frac{1}{x^k}\right) + x^{1-k} \\ -xk + x + k \leq x^{1-k}\end{aligned}$$

But now this is evident. We have Bernoulli's inequality, since $x^{1-k} = (1 + (x-1))^{1-k} \geq 1 + (x-1)(1-k) = x + k - xk$. Equality holds only where $x = 1$ or $n = 2$.

A number of formulations of this idea of equating variables have recently surfaced in various forums; it is up to the reader to choose and learn the nuances of his favorite. Although it is probably not unique, the author's opinion is that our presentation here (Karamata's majorization inequality followed by Bernoulli's inequality) is easily seen to meet the burden of rigor without the woe of undue pedantry.

[7] This $n-1$ equal value principle is particularly useful. If a differentiable function has a single inflexion point and is evaluated at n arbitrary reals with a fixed sum, any minimum or maximum must occur where some $n-1$ variables are equal.

23. (Darij Grinberg) Show that for all positive reals a, b, c,

$$\frac{\sqrt{b+c}}{a} + \frac{\sqrt{c+a}}{b} + \frac{\sqrt{a+b}}{c} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}$$

Solution 1. By Cauchy, we have $\sqrt{(a+b)(a+c)} \geq a + \sqrt{bc}$. Now,

$$\begin{aligned} \sum_{cyc} \frac{\sqrt{b+c}}{a} &\geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}} \\ \iff \sum_{cyc} \frac{b+c}{a}\sqrt{(a+b)(a+c)} &\geq 4(a+b+c) \end{aligned}$$

Substituting our result from Cauchy, it would suffice to show

$$\sum_{cyc} (b+c)\frac{\sqrt{bc}}{a} \geq 2(a+b+c)$$

WLOG $a \geq b \geq c$, implying $b+c \leq c+a \leq a+b$ and $\frac{\sqrt{bc}}{a} \leq \frac{\sqrt{ca}}{b} \leq \frac{\sqrt{ab}}{c}$. Hence, by Chebyshev and AM-GM,

$$\begin{aligned} \sum_{cyc} (b+c)\frac{\sqrt{bc}}{a} &\geq \frac{(2(a+b+c))\left(\frac{\sqrt{bc}}{a} + \frac{\sqrt{ca}}{b} + \frac{\sqrt{ab}}{c}\right)}{3} \\ &\geq 2(a+b+c) \end{aligned}$$

as desired.

Solution 2. Let $x = \sqrt{b+c}, y = \sqrt{c+a}, z = \sqrt{a+b}$. Then x, y, z are the sides of acute triangle XYZ (in the typical manner), since $x^2 + y^2 = a + b + 2c > a + b = z^2$. The inequality is equivalent to

$$\sum_{cyc} \frac{x}{y^2 + z^2 - x^2} \geq \frac{x^2 + y^2 + z^2}{xyz}$$

Recalling that $y^2 + z^2 - x^2 = 2yz\cos(X)$, we reduce this to the equivalent

$$\sum_{cyc} \frac{x^2}{\cos(X)} \geq 2(x^2 + y^2 + z^2)$$

WLOG, we have $x \geq y \geq z$, implying $\frac{1}{\cos(X)} \geq \frac{1}{\cos(Y)} \geq \frac{1}{\cos(Z)}$, so that applying Chebyshev to the left reduces the desired to proving that the sum of the reciprocals of the cosines is at least 6. By AM-HM,

$$\frac{1}{\cos(X)} + \frac{1}{\cos(Y)} + \frac{1}{\cos(Z)} \geq \frac{9}{\cos(X) + \cos(Y) + \cos(Z)}$$

But recall from triangle geometry that $\cos(X)+\cos(Y)+\cos(Z) = 1+\frac{r}{R}$ and $R \geq 2r$. The desired is now evident.

By now, the first solution should look fairly standard. The ideas of the second solution are new, however, and merit further comment. The presence of radicals in inequalities limit our pure algebraic manipulation unless we are willing to square them out. In the first solution, we managed with Cauchy and AM-GM, but the second solution shows that it is also possible to simply define new variables to clear them out. It is important to recognize and interpret the implicit restrictions on any new variables, in this case they were sides of an acute triangle. Finally, it is sometimes helpful to look at the geometric interpretations of algebraic expressions. From our triangle we conjure the law of Cosines, which can be rearranged to give an alternate means of writing the denominators on the left. The introduction of cosine is hardly a problem, in fact it should probably be regarded favorably, since we know that cosine is concave over $[0, \pi/2]$ and thus probably workable by Jensen. But, continuing with triangle geometry, we see that this fact is not needed because the desired relation follows from well-known triangle formulae.

24. Show that for all positive numbers $x_1, \ldots, x_n$,

$$\frac{x_1^3}{x_1^2+x_1x_2+x_2^2}+\frac{x_2^3}{x_2^2+x_2x_3+x_3^2}+\cdots+\frac{x_n^3}{x_n^2+x_nx_1+x_1^2} \geq \frac{x_1+\cdots+x_n}{3}$$

Solution. Observe that $0 = (x_1-x_2)+(x_2-x_3)+\cdots+(x_n-x_1) = \sum_{i=1}^n \frac{x_i^3-x_{i+1}^3}{x_i^2+x_ix_{i+1}+x_{i+1}^2}$ (where $x_{n+1} = x_1$). Hence,

$$\sum_{i=1}^n \frac{x_i^3}{x_i^2+x_ix_{i+1}x_{i+1}^2} = \frac{1}{2}\sum_{i=1}^n \frac{x_i^3+x_{i+1}^3}{x_i^2+x_ix_{i+1}+x_{i+1}^2}$$

But now $a^3+b^3 \geq \frac{1}{3}a^3+\frac{2}{3}a^2b+\frac{2}{3}ab^2+\frac{1}{3}b^3 = \frac{1}{3}(a+b)(a^2+ab+b^2)$. Hence,

$$\frac{1}{2}\sum_{i=1}^n \frac{x_i^3+x_{i+1}^3}{x_i^2+x_ix_{i+1}+x_{i+1}^2} \geq \frac{1}{2}\sum_{i=1}^n \frac{x_i+x_{i+3}}{3} = \frac{1}{3}\sum_{i=1}^n x_i$$

as desired.

This problem is difficult until the solver realizes that the cyclicly written inequality is actually symmetric. After rewriting the desired relation, all that remains is a short isolated fudging argument.

25. Let a, b, c be positive reals such that $a+b \geq c$; $b+c \geq a$; and $c+a \geq b$, we have

$$2a^2(b+c)+2b^2(c+a)+2c^2(a+b) \geq a^3+b^3+c^3+9abc$$

Solution. To handle the constraints, put $a = y+z, b = z+x, c = x+y$ with $x, y, z \geq 0$. Now, the left hand side becomes $4x^3+4y^3+4z^3+10x^2(y+z)+10y^2(z+$

$x) + 10z^2(x+y) + 24xyz$ while the right hand side becomes $2x^3 + 2y^3 + 2z^3 + 12x^2(y + z) + 12y^2(z+x) + 12z^2(x+y) + 18xyz$. The desired is seen to be equivalent to $x^3 + y^3 + z^3 + 3xyz \geq x^2(y+z) + y^2(z+x) + z^2(x+y)$, which is Schur's inequality. Equality holds where $x = y = z$, which gives $(a, b, c) = (t, t, t)$, or when two of x, y, z are equal and the third is 0, which gives $(a, b, c) \in \{(2t, t, t), (t, 2t, t), (t, t, 2t)\}$.

Certainly, the free-variable inequality does not hold; one merely sets $b = c \to 0$. Inspecting, equality holds for $(a, b, c) = (t, t, t)$ and $(2t, t, t)$. The devious author thought he could hide the nontrivial equality cases! In other words, any cavalier use of AM-GM, Jensen, or Cauchy with unique equality case $a = b = c$ will immediately falsify the problem. There aren't many famous inequalities in our toolbox with nontrivial equality cases, and so after implicitly removing the restrictions, it is hardly surprising that the problem quickly reduces to Schur's.

26. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}} + \frac{b}{\sqrt{2c^2+2a^2-b^2}} + \frac{c}{\sqrt{2a^2+2b^2-c^2}} \geq \sqrt{3}$$

Solution 1. Again write $a = y + z$, $b = z + x$, and $c = x + y$, noting that x, y, z are positive. (Triangles are generally taken to be non-degenerate when used in inequalities.) We have

$$\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}} = \sum_{cyc} \frac{y+z}{\sqrt{4x^2+4xy+4xz+y^2+z^2-2yz}}$$

Consider the convex function $f(x) = \frac{1}{\sqrt{x}}$. (As we shall see, Jensen almost always provides a tractable means of eliminating radicals from inequalities.) Put $x+y+z = 1$. We have

$$\sum_{cyc}(y+z)f\left(4x^2+4xy+4xz+y^2+z^2-2yz\right) \geq$$

$$((y+z)+(z+x)+(x+y))\, f\left(\frac{\sum_{cyc}(y+z)\left(4x^2+4xy+4xz+y^2+z^2-2yz\right)}{(y+z)+(z+x)+(x+y)}\right)$$

$$= \frac{2\sqrt{2}}{\sqrt{\sum_{cyc} 4x^2(y+z) + (4xy^2+4xyz) + (4xyz+4xz^2) + y^3 + z^3 - y^2z - yz^2}}$$

Noting that $\sum_{cyc} 4x^2(y+z) + (4xy^2+4xyz) + (4xyz+4xz^2) + y^3 + z^3 - y^2z - yz^2 = \sum_{cyc} 2x^3 + 7x^2(y+z) + 8xyz$,

$$8(x+y+z)^3 \geq 3\sum_{cyc} 2x^3 + 7x^2(y+z) + 8xyz$$

$$\iff \sum_{sym} 4x^3 + 24x^2y + 8xyz \geq \sum_{sym} 3x^3 + 21x^2y + 12xyz$$

$$\iff 2x^3 + 2y^3 + 2z^3 + 3\left(x^2(y+z) + y^2(z+x) + z^2(x+y)\right) \geq 24xyz$$

which follows by AM-GM. As a follow up on an earlier mentioned connection, oberserve the similarity of the above application of Jensen and the following inequality (which follows by Hölder's inequality)

$$\left(\sum_i \alpha_i\beta_i\right)\left(\sum_i \alpha_i\frac{1}{\sqrt{\beta_i}}\right)^2 \geq \left(\sum_i \alpha_i\right)^3$$

Solution 2 [by Darij Grinberg.] Let ABC be a triangle of side lengths a, b, c in the usual order. Denote by m_a, m_b, m_c the lengths of the medians from A, B, C respectively. Recall from triangle goemetry that $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, so that we need only show $\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq 2\sqrt{3}$. But a triangle with side lengths m_a, m_b, m_c, in turn, has medians of length $\frac{3a}{4}$, $\frac{3b}{4}$, and $\frac{3c}{4}$. The desired inequality is therefore equivalent to $\frac{\frac{4}{3}m_a}{a} + \frac{\frac{4}{3}m_b}{b} + \frac{\frac{4}{3}m_c}{c} \geq 2\sqrt{3}$ where we refer to the new triangle ABC. Recalling that $\frac{2}{3}m_a = AG$, where G is the centroid, the desired is seen to be equivalent to the geometric inequality $\frac{AG}{a} + \frac{BG}{b} + \frac{CG}{c} \geq \sqrt{3}$. But we are done as we recall from triangle geometry that $\frac{AM}{a} + \frac{BM}{b} + \frac{CM}{c} \geq \sqrt{3}$ holds for any point inside triangle ABC.[8]

As with example 22, we give two solutions of different flavors. A completely algebraic solution, and a solution that draws heavily on geometry. We elaborate on the former. Perhaps this is the first time the reader has seen Jensen used to clear radicals from denominators. This is actually a very common idea. In time, it will be recognized as an obvious consequence of the convexity of $f(x) = x^c$ for $c < 0$. Precisely the same solution can be given by Hölder's inequality as the subtle connection between Jensen and Hölder resurfaces. However, c can assume a continuum of values in the Jensen solution, and Hölder can be used on several distinct sequences as we shall see shortly.

27. (IMO 99/2) For $n \geq 2$ a fixed positive integer, find the smallest constant C such that for all nonnegative reals $x_1, \ldots, x_n$,

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2 + x_j^2) \leq C\left(\sum_{i=1}^n x_i\right)^4$$

Solution. The answer is $C = \frac{1}{8}$, which is obtained when any two x_i are non-zero and equal and the rest are 0. Observe that by AM-GM,

$$\begin{aligned}(x_1 + \cdots + x_n)^4 &= \left(\sum_{i=1}^n x_i^2 + 2\sum_{1\leq i<j\leq n} x_ix_j\right)^2 \\ &\geq 4\left(\sum_{i=1}^n x_i^2\right)\left(2\sum_{1\leq i<j\leq n} x_ix_j\right)\end{aligned}$$

[8] For a complete proof of this last inequality, see http://www.mathlinks.ro/Forum/viewtopic.php?t=21016 post #14.

$$= 8 \sum_{1 \le i < j \le n} x_i x_j \sum_{k=1}^{n} x_k^2$$

But $x_1^2 + \cdots + x_n^2 \ge x_i^2 + x_j^2$ with equality iff $x_k = 0$ for all $k \ne i, j$. It follows that

$$(x_1 + \cdots + x_n)^4 \ge 8 \sum_{1 \le i < j \le n} x_i x_j \left(x_i^2 + x_j^2\right)$$

as desired.

When written forward, this solution may seem rather arbitrary, but it is actually fairly well motivated. Taking the extremal value $n = 2$, we can rewrite the inequality as

$$0 \le a^4 + (4 - 1/c)a^3 b + 6a^2 b^2 + (4 - 1/c)ab^3 + b^4 = (a-b)^4 + (8 - 1/c)ab(a^2 + b^2)$$

Which is a good reason to conjecture $C = 1/8$. Then, taking $x_n = 0$, we show that the case $n = k - 1$ is a special case of the stronger inequality where $n = k$, so that for $n > 2$ we have $C \ge 1/8$.

We should always ask what a reasonable yet provable answer is. In this case, if the minimal C is greater than $1/8$ for some n, then there must be an even stronger case than $x_1 = x_2 = 1, x_3 = \cdots = x_n = 0$. That seems unreasonable, being as though our current equality case is already highly nontrivial, so we proceed to establish our conjectured answer.

28. Show that for nonnegative reals a, b, c,

$$2a^6 + 2b^6 + 2c^6 + 16a^3b^3 + 16b^3c^3 + 16c^3a^3 \ge 9a^4(b^2 + c^2) + 9b^4(c^2 + a^2) + 9c^4(a^2 + b^2)$$

Solution 1. Consider

$$\sum_{cyc} (a-b)^6 = \sum_{cyc} a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \ge 0$$

and

$$\sum_{cyc} ab(a-b)^4 = \sum_{cyc} a^5b - 4a^4b^2 + 6a^3b^3 - 4a^2b^4 + ab^5 \ge 0$$

Adding six times the latter to the former yields the desired result.

Solution 2. We shall prove $a^6 - 9a^4b^2 + 16a^3b^3 - 9a^2b^4 + b^6 \ge 0$. We have

$$\begin{aligned} a^6 - 2a^3b^3 + b^6 &= (a^3 - b^3)^2 \\ &= \left((a-b)(a^2 + ab + b^2)\right)^2 \\ &\ge (a-b)^2(3ab)^2 = 9a^4b^2 - 18a^3b^3 + 9a^2b^4 \end{aligned}$$

As desired. The result now follows from adding this lemma cyclicly. The main difficulty with this problem is the absence of a^5b terms on the right and also the presence of

a^4b^2 terms on the right - contrary to where Schur's inequality would generate them. Evidently AM-GM is too weak to be applied directly, since $a^6 + 2a^3b^3 \geq 3a^4b^2$ cannot be added symmetrically to deduce the problem. By introducing the factor $(a-b)^2$, however, we weight the AM-GM by a factor which we "know" will be zero at equality, thereby increasing its sharpness.

29. Let $0 \leq a, b, c \leq \frac{1}{2}$ be real numbers with $a + b + c = 1$. Show that

$$a^3 + b^3 + c^3 + 4abc \leq \frac{9}{32}$$

Solution. Let $f(a, b, c) = a^3 + b^3 + c^3 + 4abc$ and $g(a, b, c) = a + b + c = 1$. Because f and g are polynomials, they have continuous first partial derivatives. Moreover, the gradient of g is never zero. Hence, by the theorem of **Lagrange Multipliers**, any extrema occur on the boundary or where $\nabla f = \lambda \nabla g$ for suitable scalars λ. As $\nabla f =< 3a^2 + 4bc, 3b^2 + 4ca, 3c^2 + 4ab >$ and $\nabla g =< 1, 1, 1 >$, we have

$$\begin{aligned} \lambda &= 3a^2 + 4bc \\ &= 3b^2 + 4ca \\ &= 3c^2 + 4ab \\ g(a, b, c) &= a + b + c = 1 \end{aligned}$$

We have $3a^2 + 4bc = 3b^2 + 4ca$ or $(a - b)(3a + 3b - 4c) = (a - b)(3 - 7c) = 0$ for any permutation of a, b, c. Hence, without loss of generality, $a = b$. Now, $3a^2 + 4ac = 3c^2 + 4a^2$ and $a^2 - 4ac + 3c^2 = (a - c)(a - 3c) = 0$. The interior local extrema therefore occur when $a = b = c$ or when two of $\{a, b, c\}$ are three times as large as the third. Checking, we have $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 7/27 < 13/49 = f(\frac{1}{7}, \frac{3}{7}, \frac{3}{7})$. Recalling that $f(a, b, c)$ is symmetric in a, b, c, the only boundary check we need is $f(\frac{1}{2}, t, \frac{1}{2} - t) \leq \frac{9}{32}$ for $0 \leq t \leq \frac{1}{2}$. We solve

$$\begin{aligned} h(t) &= f\left(\frac{1}{2}, t, \frac{1}{2} - t\right) \\ &= \frac{1}{8} + t^3 + \left(\frac{1}{2} - t\right)^3 + 2t\left(\frac{1}{2} - t\right) \\ &= \frac{1}{4} + \frac{t}{4} - \frac{t^2}{2} \end{aligned}$$

$h(t)$ is $\frac{1}{4}$ at either endpoint. Its derivative $h'(t) = \frac{1}{4} - t$ is zero only at $t = \frac{1}{4}$. Checking, $h(\frac{1}{4}) = f(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = \frac{9}{32}$. Since $h(t)$ has a continuous derivative, we are done. (As a further check, we could observe that $h''(t) = -1 < 0$, which guarantees that $h(\frac{1}{4})$ is a local maximum.)

Usage Note. *The use of Lagrange Multipliers in any solution will almost certainly draw hostile review, in the sense that the tiniest of errors will be grounds for null* *marks. If you consider multipliers on Olympiads, be diligent and provide explicit, kosher remarks about the continuous first partial derivatives of both $f(x_1, \ldots, x_n)$ and the constraint $g(x_1, \ldots, x_n) = k$, as well as $\nabla g \neq \mathbf{0}$, before proceeding to solve the system $\nabla f = \lambda \nabla g$. The main reason this approach is so severely detested is that, given sufficient computational fortitude (if you are able to sort through the relevant algebra and Calculus), it can and will produce a complete solution. The example provided here is included for completeness of instruction; typical multipliers solutions will not be as clean or painless.*[9]

30. (Vasile Cirtoaje) Let $p \geq 2$ be a real number. Show that for all nonnegative reals a, b, c,

$$\sqrt[3]{\frac{a^3 + pabc}{1+p}} + \sqrt[3]{\frac{b^3 + pabc}{1+p}} + \sqrt[3]{\frac{c^3 + pabc}{1+p}} \leq a + b + c$$

Solution. By Hölder,

$$\left(\sum_{cyc} \sqrt[3]{\frac{a^3 + pabc}{1+p}}\right)^3 \leq \left(\sum_{cyc} \frac{1}{1+p}\right)\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 + pbc\right)$$

But $a^2 + b^2 + c^2 \geq ab + bc + ca$ (proven by AM-GM, factoring, or a number of other methods) implies that

$$\sum_{cyc} a^2 + pbc \leq (p+1)\sum_{cyc} \frac{a^2 + 2bc}{3} = \frac{p+1}{3}(a+b+c)^2$$

From which we conclude

$$\left(\sum_{cyc} \sqrt[3]{\frac{a^3 + pabc}{1+p}}\right)^3 \leq (a+b+c)^3$$

as desired.

This problem illustrates one feature that Hölder's inequality holds over Jensen's inequality: we can use unequal sequences. We base our selection of sequences on the factorization of the arguments within each cube root - $\frac{a(a^2+pbc)}{1+p}$ - which is especially attractive because $p = 2$ then carries special significance as the value of p where we obtain the identity $a^2 + b^2 + c^2 + 2(bc + ca + ab) = (a+b+c)^2$.

31. Let a, b, c be real numbers such that $abc = -1$. Show that

$$a^4 + b^4 + c^4 + 3(a+b+c) \geq \frac{a^2}{b} + \frac{a^2}{c} + \frac{b^2}{c} + \frac{b^2}{a} + \frac{c^2}{a} + \frac{c^2}{b}$$

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[9] Just how painful can the calculations get? Most multipliers solutions will tend to look more like http://documents.wolfram.com/mathematica/Demos/Notebooks/InequalityProof.html than this solution.

Solution. First we homogenize, obtaining $a^4+b^4+c^4+a^3(b+c)+b^3(c+a)+c^3(a+b)-3abc(a+b+c)\geq 0$. As this is homogenous in the fourth degree, we can scale a,b,c by any real k and hence may now ignore $abc=-1$. Equality holds at $a=b=c=1$, but also at $a=b=1, c=-2$, $a=1, b=0, c=-1$, and a number of unusual locations with the commonality that $a+b+c=0$. Indeed, $c=-a-b$ is a parametric solution, and we discover the factorization $(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)\geq 0$. (We are motivated to work with factorizations because there are essentially no other inequalities with $a+b+c=0$ as an equality condition.)

32. (MOP 2003) Show that for all nonnegative reals a,b,c,

$$\begin{aligned} a^4(b^2+c^2)+b^4(c^2+a^2)+c^4(a^2+b^2) &+ \\ 2abc(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b-a^3-b^3-c^3-3abc) & \\ &\geq 2a^3b^3+2b^3c^3+2c^3a^3 \end{aligned}$$

Solution. As was suggested by the previous problem, checking for equality cases is important when deciding how to solve a problem. We see that setting $a=b$ produces equality. As the expression is symmetric, this certainly implies that $b=c$ and $c=a$ are equality cases. Hence, if $P(a,b,c)$ is the difference LHS - RHS, then $(a-b)(b-c)(c-a)|P(a,b,c)$. Obviously, if the problem is going to be true, $(a-b)$ must be a double root of P, and accordingly we discover the factorization $P(a,b,c)=(a-b)^2(b-c)^2(c-a)^2$.

The result illustrated above was no accident. If $(x-y)$ divides a symmetric polynomial $P(x,y,z)$, then $(x-y)^2$ divides the same polynomial. If we write $P(x,y,z)=(x-y)Q(x,y,z)$, then $(x-y)Q(x,y,z)=P(x,y,z)=P(y,x,z)=(y-x)Q(y,x,z)$, which gives $Q(x,y,z)=-Q(y,x,z)$. Hence $Q(x,x,z)=0$, and $(x-y)$ also divides $Q(x,y,z)$.

33. (Cezar Lupu) Let a,b,c be positive reals such that $a+b+c+abc=4$. Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\frac{\sqrt{2}}{2}\cdot(a+b+c)$$

Solution. By Cauchy

$$\left(\sum_{cyc}a\sqrt{b+c}\right)\left(\sum_{cyc}\frac{a}{\sqrt{b+c}}\right)\geq(a+b+c)^2$$

But, also by Cauchy,

$$\sqrt{(a+b+c)\,(a(b+c)+b(c+a)+c(a+b))}\geq\sum_{cyc}a\sqrt{b+c}$$

Hence,

$$\sum_{cyc}\frac{a}{\sqrt{b+c}}\geq\frac{\sqrt{2}}{2}\cdot(a+b+c)\cdot\sqrt{\frac{a+b+c}{ab+bc+ca}}$$

And we need only show $a+b+c \geq ab+bc+ca$. Schur's inequality for $r=1$ can be expressed as $\frac{9abc}{a+b+c} \geq 4(ab+bc+ca)-(a+b+c)^2$. Now, we suppose that $ab+bc+ca > a+b+c$, and have

$$\begin{aligned} \frac{9abc}{a+b+c} &\geq 4(ab+bc+ca)-(a+b+c)^2 \\ &> (a+b+c)\left(4-(a+b+c)\right) = abc(a+b+c) \end{aligned}$$

Hence, $a+b+c < 3$. But then $abc < 1$, which implies $4 = a+b+c+abc < 4$. Contradiction, as desired.

Our proof of the result $a+b+c+abc = 4 \Longrightarrow a+b+c \geq ab+bc+ca$ is necessarily a bit magical, but it indicates another way to successfully use given constraints besides clever substitution or homogenization. Strange relations passed off as equations can sometimes be interpreted as simple inequalities.

34. (Iran 1996) Show that for all positive real numbers a, b, c,

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \geq \frac{9}{4}$$

Solution. *Fearless courage is the foundation of all success.*[10] When everything else fails, return to the sure-fire strategy of clearing all denominators. In this case, we obtain

$$4(a+b)^2(b+c)^2(c+a)^2(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) = \\ \sum_{sym} 4a^5b+8a^4b^2+10a^4bc+6a^3b^3+52a^3b^2c+16a^2b^2c^2$$

on the left, and on the right,

$$9(a+b)^2(b+c)^2(c+a)^2 = \\ \sum_{sym} 9a^4b^2+9a^4bc+9a^3b^3+54a^3b^2c+15a^2b^2c^2$$

Canceling like terms, we seek

$$\sum_{sym} 4a^5b-a^4b^2+a^4bc-3a^3b^3-2a^3b^2c+a^2b^2c^2$$

Sure enough, this is true, since $\frac{3a^5b+ab^5}{4} \geq a^4b^2$ and $\frac{a^4b^2+a^2b^4}{2} \geq a^3b^3$ by AM-GM, and $abc\left(a^3+b^3+c^3-a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc\right) \geq 0$ by Schur.

[10] Found on a fortune cookie by Po-Ru Loh while grading an inequality on 2005 Mock IMO Day 2 that was solved by brutal force.

In fact, this is the only known solution to this inequality. And it really did appear on a national Olympiad! This was probably a one-of-a-kind occurrence, however, because problems whose only solution is uninspiring dumbassing are not especially popular nowadays. Nevertheless, one should never categorically discount expansion unless it is absolutely ridiculous.

35. (Japan 1997) Show that for all positive reals a, b, c,

$$\frac{(a+b-c)^2}{(a+b)^2+c^2} + \frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} \geq \frac{3}{5}$$

Solution. Put $a+b+c=3$ so that equality will hold at $a=b=c=1$ and suppose that there exists some k for which

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} = \frac{(3-2a)^2}{(3-a)^2+a^2} \geq \frac{1}{5} + ka - k$$

for all positive a, b, c; such an inequality would allow us to add cyclicly to deduce the desired inequality. As the inequality is parametrically contrived to yield equality where $a=1$, we need to find k such that $a=1$ is a double root. At $a=1$, the derivative on the left is $\frac{(2(3-2a)\cdot-2)((3-a)^2+a^2)-((3-2a)^2)(2(3-a)\cdot-1+2a)}{((3-a)^2+a^2)^2} = \frac{-18}{25}$. The derivative on the right is k, so we set $k = \frac{-18}{25}$. But for this k we find

$$\begin{aligned}(3-2a)^2 - \left(\frac{1}{5} + ka - k\right)\left((3-a)^2+a^2\right) &= \frac{18}{25} - \frac{54a^2}{25} + \frac{36a^3}{25} \\ &= \frac{18}{25}(a-1)^2(2a+1) \geq 0\end{aligned}$$

as desired. Alternatively, we could have used AM-GM to show $a^3+a^3+1 \geq 3a^2$. As hinted at by a previous problem, inequalities are closely linked to polynomials with roots of even multiplicity. The isolated manipulation idea used in this solution offers a completely different approach to the inequalities which work with every term.

36. (MOP 02) Let a, b, c be positive reals. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3$$

Solution. Suppose that there exists some r such that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq \frac{3a^r}{a^r+b^r+c^r}$$

We could sum the inequality cyclicly to deduce what we want. Since equality holds at $a=b=c=1$, we use derivatives to find a suitable r. At the said equality case, on the left, the partial derivative with respect to a is $\frac{2}{3}$, while the same derivative on the right is $\frac{2}{3}r$. Equating the two we have $r = 1$. (This is necessary since otherwise the inequality will not hold for either $a = 1 + \epsilon$ or $a = 1 - \epsilon$.)[11] Now,

$$\begin{aligned} \frac{3a}{a+b+c} &\leq \frac{3a}{3\sqrt[3]{a \cdot \left(\frac{b+c}{2}\right)^2}} \\ &= \frac{a^{\frac{2}{3}}}{\left(\frac{b+c}{2}\right)^{\frac{2}{3}}} \\ &= \left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \end{aligned}$$

by AM-GM, as desired.

37. (Mildorf) Let $n \geq 2$ be an integer. Prove that for all reals $a_1, a_2, \ldots, a_n > 0$ and reals $p, k \geq 1$,

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{a_1^p + a_2^p + \cdots + a_n^p}\right)^k \geq \frac{a_1^k + a_2^k + \cdots + a_n^k}{a_1^{pk} + a_2^{pk} + \cdots + a_n^{pk}}$$

where inequality holds iff $p = 1$ or $k = 1$ or $a_1 = a_2 = \cdots = a_n$, flips if instead $0 < p < 1$, and flips (possibly again) if instead $0 < k < 1$.

Solution. Taking the kth root of both sides, we see that the inequality is equivalent to

$$\sum_{i=1}^{n} \sqrt[k]{\frac{a_i^k}{a_1^k + a_2^k + \cdots + a_n^k}} \geq \sum_{i=1}^{n} \sqrt[k]{\frac{a_i^{pk}}{a_1^{pk} + a_2^{pk} + \cdots a_n^{pk}}}$$

WLOG, suppose that $a_1 \geq a_2 \geq \cdots \geq a_n$. We prove a lemma. Let $S_i = \frac{a_i^p}{a_1^p + \cdots + a_n^p}$ and $T_i = \frac{a_i^q}{a_1^q + \cdots + a_n^q}$ for $i = 1, 2, \ldots, n$ where $0 < q < p$. Then the sequence $S_1, S_2, \ldots, S_n$ majorizes the sequence $T_1, T_2, \ldots, T_n$.

To prove the claim, we note that $S_1 \geq \cdots \geq S_n$ and $T_1 \geq \cdots \geq T_n$ and have, for $m \leq n$,

$$\begin{aligned} \sum_{i=1}^{m} S_i &\geq \sum_{i=1}^{m} T_i \\ \iff (a_1^p + \cdots + a_m^p)(a_1^q + \cdots + a_n^q) &\geq (a_1^q + \cdots + a_m^q)(a_1^p + \cdots + a_n^p) \\ \iff (a_1^p + \cdots + a_m^p)\left(a_{m+1}^q + \cdots + a_n^q\right) &\geq (a_1^q + \cdots + a_m^q)\left(a_{m+1}^p + \cdots + a_n^p\right) \\ \iff \sum_{(i,j)\,|\,\{1 \leq i \leq m < j \leq n\}} a_i^p a_j^q - a_i^q a_j^p &\geq 0 \end{aligned}$$

[11] Actually, even this is a special case of the general sense that the convexity of one side must exceed the convexity of the other. More precisely, we have the following result: Let f and g functions over the domain D with continuous partial derivatives. If $f(\nu) \geq g(\nu)$ for all $\nu \in D$, then at every equality case $\nu_{\mathbf{0}}$, $\nabla(f-g)(\nu_{\mathbf{0}}) = \mathbf{0}$ and every component of $\nabla^2(f-g)(\nu_{\mathbf{0}})$ is nonnegative.

Which is obvious. In particular, $m = n$ is the equality case, and the claim is established. But now the desired is a direct consequence of the Majorization inequality applied to the sequences in question and the function $f(x) = \sqrt[k]{x}$.

38. (Vasile Cirtoaje) Show that for all real numbers a, b, c,

$$(a^2 + b^2 + c^2)^2 \geq 3\left(a^3b + b^3c + c^3a\right)$$

Solution. We will be content to give the identity

$$(a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a) = \frac{1}{2}\sum_{cyc}\left(a^2 - 2ab + bc - c^2 + ca\right)^2$$

Any Olympiad partipant should be comfortable constructing various inequalities through well-chosen squares. Here, we could certainly have figured we were summing the square of a quadratic that is 0 when $a = b = c$ such that no term a^2bc is left uncancelled. A good exercise is to show that equality actually holds iff $a = b = c$ or, for some cyclic permutation, $a : b : c \equiv \sin^2\left(\frac{4\pi}{7}\right) : \sin^2\left(\frac{2\pi}{7}\right) : \sin^2\left(\frac{\pi}{7}\right)$.

39. (Anh-Cuong) Show that for all nonnegative reals a, b, c,

$$a^3 + b^3 + c^3 + 3abc \geq ab\sqrt{2a^2 + 2b^2} + bc\sqrt{2b^2 + 2c^2} + ca\sqrt{2c^2 + 2a^2}$$

Solution. Upon observing that this inequality is stronger than Schur's inequality for $r = 1$, we are inspired to prove a sharp lemma to eliminate the radical. Knowing that $\sqrt{2x^2 + 2y^2} \geq x + y \geq \frac{2xy}{x+y}$, we seek a combination of the latter two that exceeds the former. We find

$$\frac{3x^2 + 2xy + 3y^2}{2(x + y)} \geq \sqrt{2x^2 + 2y^2}$$

This follows from algebra, since $\left(3x^2 + 2xy + 3y^2\right)^2 = 9x^4 + 12x^3y + 22x^2y^2 + 12xy^3 + 9y^4 \geq 8x^4 + 16x^3y + 16x^2y^2 + 16xy^3 + 8y^4 = 4(x + y)^2(2x^2 + 2y^2)$, so that $(3x^2 + 2xy + 3y^2)^2 - 4(x + y)^2(2x^2 + 2y^2) = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = (x - y)^4 \geq 0$. Now,

$$\sum_{cyc} ab\sqrt{2a^2 + 2b^2} \leq \sum_{cyc}\frac{(3a^2 + 2ab + 3b^2)ab}{2(a + b)}$$

So it would suffice to show

$$\begin{aligned}
\sum_{cyc} a(a - b)(a - c) &= \sum_{cyc}(a^3 + abc - ab(a + b)) \\
&\geq \sum_{cyc}\frac{(3a^2 + 2ab + 3b^2)ab}{2(a + b)} - ab(a + b) \\
&= \sum_{cyc}\frac{3a^3b + 2a^2b^2 + 3ab^3 - 2a^3b - 4a^2c^2 - 2ab^3}{2(a + b)} \\
&= \sum_{cyc}\frac{ab(a - b)^2}{2(a + b)}
\end{aligned}$$

But

$$\sum_{cyc}(b+c-a)(b-c)^2 = 2\sum_{cyc} a(a-b)(a-c)$$

so that the desired is

$$\sum_{cyc}\left(b+c-a-\frac{bc}{b+c}\right)(b-c)^2 \geq 0$$

which is evident, since without loss of generality we may assume $a \geq b \geq c$ and find

$$\begin{aligned} \left(a+b-c-\frac{ab}{a+b}\right)(a-b)^2 &\geq 0 \\ \left(c+a-b-\frac{ac}{a+c}\right)\left((a-c)^2-(b-c)^2\right) &\geq 0 \\ \left(b+c-a-\frac{bc}{b+c}\right)(b-c)^2 + \left(c+a-b-\frac{ac}{a+c}\right)(b-c)^2 &\geq 0 \end{aligned}$$

The key to this solution was the sharp upper bound on the root-mean-square. At first glance our lemma seems rather arbitrary and contrived. Actually, it is a special case of a very sharp bound on the two variable power mean that I have conjectured and proved.

Mildorf's Lemma 1 *Let $k \geq -1$ be an integer. Then for all positive reals a and b,*

$$\frac{(1+k)(a-b)^2+8ab}{4(a+b)} \geq \sqrt[k]{\frac{a^k+b^k}{2}}$$

with equality if and only if $a=b$ or $k=\pm 1$, where the power mean $k=0$ is interpreted to be the geometric mean $\sqrt{ab}$. Moreover, if $k<-1$, then the inequality holds in the reverse direction, with equality if and only if $a=b$.

Usage Note. *In early November 2005, I proved an extension of this lemma to additional values of k.[12] Thus, you may rest assured that the result stated above is true. I have since had the proof published online in the new journal* ***Mathematical Reflections****. The article is posted at*

http://reflections.awesomemath.org/2006_2/2006_2_sharpbound.pdf

I have also posted a copy at my MIT site as "ASharpBound.pdf." However, the proof is rather difficult (or at least so I think, being as though it took me nearly half a year) and the lemma is far from mainstream. Thus, should you require it on an Olympiad, you should prove it for whatever particular value of k you are invoking. This is not terribly difficult if k is a small integer. One simply takes the kth power of both sides and factors the difference of the two sides as $(a-b)^4 \cdot P(a,b)$, etc.

[12] In particular, the inequality holds for all k in $(-\infty,-1), \{-1,0,1\}, (1,3/2], [2,\infty)$ with the signs $\leq, \geq, \leq, \geq$ respectively, with equality iff $a=b$ or $k=\pm 1$.

40. For $x \geq y \geq 1$, prove that

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}$$

Solution. By observation, equality holds when $y = 1$ and when $x = y$. Combining this with the restriction, it makes sense to write $x = y+a$ and $y = 1+b$ where $a, b \geq 0$. Now we can write

$$\begin{aligned} \frac{x-y}{\sqrt{x+y}} + \frac{y-1}{\sqrt{y+1}} + \frac{1-x}{\sqrt{1+x}} &\geq 0 \\ \iff \frac{a}{\sqrt{2+a+2b}} + \frac{b}{\sqrt{2+b}} \geq \frac{a+b}{\sqrt{2+a+b}} \end{aligned}$$

But this is evident by Jensen's inequality applied to the convex function $f(x) = \frac{1}{\sqrt{x}}$, since

$$\begin{aligned} af(2+a+2b) + bf(2+b) &\geq (a+b)f\left(\frac{a(2+a+2b)+b(2+b)}{a+b}\right) \\ &= (a+b)f\left(\frac{(a+b)^2+2(a+b)}{a+b}\right) \\ &= \frac{a+b}{\sqrt{2+a+b}} \end{aligned}$$

as desired.

The intended solution to this problem is very similar but far more difficult, employing Jensen (or Cauchy) in conjunction Abel summation on the variables x, y, z (1 is an arbitrary substitute for $z \leq y$.) That solution is far more difficult than ours, however, for the substitution makes the line of attack quite obvious.

41. (Vasile Cirtoaje) Show that for positive reals a, b, c,

$$\frac{1}{4a^2-ab+4b^2} + \frac{1}{4b^2-bc+4c^2} + \frac{1}{4c^2-ca+4a^2} \geq \frac{9}{7(a^2+b^2+c^2)}$$

Solution. Upon expansion, we see that it is equivalent to

$$\sum_{sym} 56a^6 - 28a^5b + 128a^4b^2 + 44a^3b^3 + \frac{95}{2}a^4bc + 31a^3b^2c - \frac{45}{2}a^2b^2c^2 \geq 0$$

We conjure up the following inequalities:

$$\sum_{sym} a^6 - 2a^5b + a^4bc \geq 0 \tag{1}$$

$$\sum_{sym} a^5b - 4a^4b^2 + 3a^3b^3 \geq 0 \quad (2)$$

$$\sum_{sym} a^4b^2 - a^4bc - a^3b^3 + 2a^3b^2c - a^2b^2c^2 \geq 0 \quad (3)$$

$$\sum_{sym} a^4bc - 2a^3b^2c + a^2b^2c^2 \geq 0 \quad (4)$$

(1) and (4) follow from Schur's inequality for $r = 4$ and $r = 1$ (multiplied by abc) respectively. (2) is the result of expanding $\sum_{cyc} ab(a-b)^4 \geq 0$, and (3) is the expanded form of the famous $(a-b)^2(b-c)^2(c-a)^2 \geq 0$. The desired now follows by subtracting 56 times (1), 84 times (2), 208 times (3), $\frac{399}{2}$ times (4), and then simple AM-GM to clear the remaining $a^2b^2c^2$.

This is about as difficult as a dumbass solution can get. A good general strategy is to work with the sharpest inequalities you can find until you reduce a problem to something obvious, starting with the most powerful (most bunched, in this case $\sum_{sym} a^6$) term and work your way down to the weak terms while keeping the most powerful term's coefficient positive. My solution to this problem starts with (1), Schur with $r = 4$ (Schur is stronger for larger r), which is almost certainly sharper than the inequality in question. Next, inequality (2) is a sharp cyclic sum to use the a^5b terms. In particular, it relates terms involving only two of the three variables. Most of the time, the only inequality that can "pull up" symmetric sums involving three variables to stronger ones involving just two is Schur, although it does so at the expense of a very strong term with only one variable. Hence, we made a logical choice. Inequality (3) is extremely sharp, and allowed us to obtain more a^4bc and a^3b^3 terms simultaneously. In particular, it was necessary to cancel the a^3b^3 terms. I'll note that this inequality is peculiar to sixth degree symmetry in three variables - it does not belong to a family of similar, nice inequalities. Finally, inequality (4), which is a handy corollary to (3), is another Schur. Every inequality we have used so far is quite sharp, and so it is no surprise that the leftovers are the comparatively weak AM-GM.

42. (USAMO 00/6) Let $n \geq 2$ be an integer and $S = \{1, 2, \ldots, n\}$. Show that for all nonnegative reals $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$,

$$\sum_{i,j \in S} \min\{a_ia_j, b_ib_j\} \leq \sum_{i,j \in S} \min\{a_ib_j, a_jb_i\}$$

Solution. We first prove a lemma:

Lemma 1 *Let $r_1, \ldots, r_n$ be nonnegative numbers, and let $x_1, \ldots, x_n$ be artbirary reals. Then*

$$\sum_{1 \leq i,j \leq n} \min(r_i, r_j)x_ix_j \geq 0$$

Proof of lemma. We assume without loss of generality that $r_1 \le r_2 \le \cdots \le r_n$. For notational convenience, define $r_0 = 0$. Then the sum in question is equal to

$$\sum_{i=1}^{n}(r_i - r_{i-1})\left(\sum_{j=i}^{n} x_j\right)^2$$

Each summand is nonnegative, so the sum is nonnegative. ▲

Now define r_i by $r_i = \frac{\max(a_i,b_i)}{\min(a_i,b_i)} - 1$ if $a_ib_i \neq 0$ and $r_i = 0$ otherwise, and define $x_i = \operatorname{sign}(a_i - b_i)\min(a_i, b_i)$ (with $\operatorname{sign}(0) = 1$). We claim that

$$\min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j) = \min(r_i, r_j)x_ix_j \qquad (*)$$

If a_i and b_i are exchanged, both sides of $(*)$ change sines. Thus, we may assume $a_i \ge b_i$ and $a_j \ge b_j$. If $b_ib_j = 0$, then both sides are 0 and we are done, so we assume further that $b_i, b_j > 0$. Then $r_i = \frac{a_i}{b_i} - 1$ and $r_j = \frac{a_j}{b_j} - 1$, while $x_i = b_i$, and $x_j = b_j$. Plugging these expressions in, we obtain

$$\begin{aligned}\min(r_i, r_j)x_ix_j &= \min\left(\frac{a_i}{b_i} - 1, \frac{a_j}{b_j} - 1\right) b_ib_j \\ &= \min(a_ib_j, a_jb_i) - b_ib_j \\ &= \min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j)\end{aligned}$$

Applying the lemma in conjunction with $(*)$, we have

$$\sum_{1\le i,j\le n}[\min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j)] = \sum_{1\le i,j\le n}\min(r_i, r_j)x_ix_j \ge 0$$

and we are done.

43. (Gabriel Dospinescu) For any $n > 2$ find the minimal value k_n such that for any positive reals $x_1, x_2, \ldots, x_n$ with product 1 we have

$$\sum_{i=1}^{n}\frac{1}{\sqrt{1 + k_nx_i}} \le n - 1$$

Solution. From $x_1 = x_2 = \cdots = x_n = 1$, it is clear that $k_n \ge \frac{2n-1}{(n-1)^2}$. We show that $k_n = \frac{2n-1}{(n-1)^2}$ is satisfactory. Let $x_i = e^{y_i}$; the condition $x_1x_2\cdots x_n = 1$ rewrites as $y_1 + y_2 + \cdots + y_n = 0$. Consider the function $f(y) = \frac{1}{\sqrt{1+k_ne^y}}$. The desired inequality is equivalent to

$$f(y_1) + \cdots + f(y_n) \le n - 1$$

We have $f'(y) = \frac{-k_ne^y}{2(1+k_ne^y)^{3/2}}$ and $f''(y) = \frac{k_ne^y(k_ne^y-2)}{4(1+k_ne^y)^{5/2}}$. Evidently, $f(y)$ has a single inflexion point at $y_* = \ln(2/k_n)$, where f is convex over $[y_*, \infty)$ and concave over

$(-\infty, y_*]$. Without loss of generality, $y_1 \geq \cdots \geq y_k \geq y_* \geq y_{k+1} \geq \cdots \geq y_n$. Then since $([y_1 + \cdots + y_k - (k-1)y_*], y_*, y_*, \ldots, y_*) \succ (y_1, \ldots, y_k)$, Karamata's majorization inequality gives

$$f(y_1) + \cdots + f(y_k) \leq (k-1)f(y_*) + f\left(y_1 + \cdots + y_k - (k-1)y_*\right) \qquad (*)$$

Jensen's inequality yields

$$(k-1)f(y_*) + f(y_{k+1}) + \cdots + f(y_n) \leq (n-1) \cdot f\left(\frac{(k-1)y_* + y_{k+1} + \cdots + y_n}{n-1}\right) \qquad (**)$$

Combining $(*)$ and $(**)$, we see that

$$f(y_1) + \cdots + f(y_n) \leq f\left(y_1 + \cdots + y_k - (k-1)y_*\right) + (n-1) \cdot f\left(\frac{(k-1)y_* + y_{k+1} + \cdots + y_n}{n-1}\right)$$

It follows that in the original inequality, we may assume that $x_1 = \cdots = x_{n-1} = x$ and $x_n = x^{1-n}$. For convenience, define $\ell = n - 1$. We now have

$$\begin{aligned} \frac{\ell}{\sqrt{1 + \frac{2\ell+1}{\ell^2}x}} + \frac{1}{\sqrt{1 + \frac{2\ell+1}{\ell^2}x^{-\ell}}} &\leq \ell \\ \iff g(x) = \frac{\ell}{\sqrt{\ell^2 + (2\ell+1)x}} + \frac{1}{\sqrt{\ell^2 + (2\ell+1)x^{-\ell}}} &\leq 1 \end{aligned}$$

Evidently $g(1) = 1$ and $\lim_{x \to 0^+} g(x) = 1$ $(***)$. Now consider

$$g'(x) = \frac{\ell(2\ell+1)}{2} \cdot \left(\frac{-1}{\left(\ell^2 + (2\ell+1)x\right)^{3/2}} + \frac{x^{-\ell-1}}{\left(\ell^2 + (2\ell+1)x^{-\ell}\right)^{3/2}}\right)$$

In light of $(***)$, it would suffice to show that $g'(x) \leq 0$ for $x \geq 1$ and that there exists some $r \in (0, 1]$ such that $g'(x) \leq 0$ for $0 < x \leq r$ and $g'(x) \geq 0$ for $r \leq x \leq 1$. Writing $x = y^3$, we have

$$\begin{aligned} g'(x) &< 0 \\ \iff \frac{-1}{\left(\ell^2 + (2\ell+1)x\right)^{3/2}} + \frac{x^{-\ell-1}}{\left(\ell^2 + (2\ell+1)x^{-\ell}\right)^{3/2}} &< 0 \\ \iff \frac{1}{\left(x^{\frac{2\ell+2}{3}}\left(\ell^2 + (2\ell+1)x^{-\ell}\right)\right)^{3/2}} &< \frac{1}{\left(\ell^2 + (2\ell+1)x\right)^{3/2}} \\ \iff y^{2\ell+2}\left(\ell^2 + (2\ell+1)y^{-3\ell}\right) &> \ell^2 + (2\ell+1)y^3 \\ \iff \left(y^{5\ell+2} - y^{3\ell}\right)\ell^2 - \left(y^{3\ell+3} - y^{2\ell+2}\right)(2\ell+1) &> 0 \\ \iff y^{2\ell+2}\left(y^{\ell+1} - 1\right)\left(y^{\ell-2}\left(y^{\ell+1} + 1\right)\ell^2 - (2\ell+1)\right) &> 0 \qquad (****) \end{aligned}$$

Since $\ell \geq 2$, $2\ell^2 \geq 2\ell + 1$, and it is clear that there exists $r \leq 1$ such that $(****)$ holds iff $y \in (0, r) \cup (1, \infty)$, or equivalently $x \in (0, r^3) \cup (1, \infty)$, as desired.

44. (Vascile Cartoaje) For any $a, b, c, d > 0$ we have

$$2(a^3+1)(b^3+1)(c^3+1)(d^3+1) \geq (1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2)$$

Solution. By Cauchy's inequality,

$$a^3+1 = \sqrt{(a^3+1)(a+1)(a^2-a+1)} \geq (a^2+1)\sqrt{a^2-a+1}$$

so it would suffice to prove that

$$2\sqrt{(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1)} \geq 1+abcd \qquad (*)$$

Now,

$$\begin{aligned} (a^2-a+1)(b^2-b+1) &\geq a^2b^2-ab+1 \\ \Longleftrightarrow a^2+2ab+b^2-a-b-a^2b-ab^2 &\geq 0 \\ \Longleftrightarrow (a+b)(a+b-1-ab) = -(a+b)(a-1)(b-1) &\geq 0 \end{aligned}$$

Thus, we may assume that a, b, c, d are on the same side of 1. Moreover, if we divide both sides of (*) by $abcd$ and substitute $x = 1/a, y = 1/b, z = 1/c, w = 1/d$, we arrive at

$$\begin{aligned} 2\sqrt{(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1)} &\geq 1+abcd \\ \Longleftrightarrow 2\sqrt{(x^2-x+1)(y^2-y+1)(z^2-z+1)(w^2-w+1)} &\geq 1+xyzw, \end{aligned}$$

so we may assume $0 < a, b, c, d \leq 1$. Furthermore, since $a^2-a+1 = (a-\frac{1}{2})^2+3/4$, if $a < 1/2$ we may exchange a with $a' = 1-a > 1/2$. Therefore, we may assume that $1/2 \leq a, b, c, d \leq 1$. Finally, consider the function $f(x) = \ln(g(x))$, where $g(x) = e^{2x} - e^x + 1$. We have $f'(x) = g'(x)/g(x)$ and $f''(x) = \frac{g''(x)g(x)-(g'(x))^2}{(g(x))^2}$. Since $g'(x) = 2e^{2x} - e^x$ and $g''(x) = 4e^{2x} - e^x$, we have $g''(x)g(x) - (g'(x))^2 = -e^x(e^{2x} - 4e^x + 1)$. It follows that $f''(x) \geq 0$ for all $x \in [\ln(1/2), 0]$. Therefore, for any $x_1, x_2, x_3, x_4 \in [\ln(1/2), 0]$, we have

$$f(x_1)+f(x_2)+f(x_3)+f(x_4) \geq 4f\left(\frac{x_1+x_2+x_3+x_4}{4}\right)$$

Exporting this result to a, b, c, d, we have

$$(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1) \geq (\sqrt{abcd} - \sqrt[4]{abcd}+1)^4$$

Letting $\alpha = \sqrt[4]{abcd}$, it suffices to prove that $2(\alpha^2-\alpha+1)^2 \geq 1+\alpha^4$. This is evident however, since the difference of the two sides is $\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 1 = (\alpha-1)^4 \geq 0$, as required.

45. (Vascile Cartoaje) Prove that the sides a, b, c of any triangle obey

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

Solution. Clear the denominators and put $a = y+z, b = z+x, c = x+y$, where $x, y, z > 0$, so that we seek

$$\sum_{cyc} 3a^2c - 2a^2b - abc = \sum_{cyc} x^3 + 2x^2y - 3xy^2 \geq 0$$

Without loss of generality, we may assume $x = \max\{x, y, z\}$. If $x = z$ or $y \geq z$, then

$$\sum_{cyc} x^2y - xy^2 = (x-y)(x-z)(y-z) \geq 0,$$

and we are done, since $\sum_{cyc} x^3 - xy^2 \geq 0$ holds trivially by AM-GM. Otherwise, $x > z > y$. Writing $y = \alpha, z = \alpha+\beta, x = \alpha+\beta+\gamma$ for $\alpha, \beta, \gamma > 0$ and expanding, we seek

$$\sum_{cyc} x^3 + 2x^2y - 3xy^2 = 2\alpha\beta^2 + \beta^3 + 2\alpha\beta\gamma - \beta^2\gamma + 2\alpha\gamma^2 + \gamma^3 \geq 0,$$

which is obvious since $\frac{\beta^3+\beta^3+\gamma^3}{3} \geq \beta^2\gamma$.

46. (George Tsintifas, Crux Mathematicorum) Prove that for any $a, b, c, d > 0$ we have the inequality

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4$$

Solution. Let $a+b+c+d = 1$, and note that

$$\begin{aligned}
&(a+b)(b+c)(c+d)(d+a) \\
&\quad = a^2c^2 + b^2d^2 + 2abcd + abc(a+b+c) \\
&\qquad +bcd(b+c+d) + cda(c+d+a) + dab(d+a+b) \\
&\quad = (ac-bd)^2 + abc(a+b+c+d) + bcd(b+c+d+a) \\
&\qquad +cda(c+d+a+b) + dab(d+a+b+c) \\
&\geq abc + bcd + cda + dab \qquad (*)
\end{aligned}$$

On the other hand, Newton's inequality gives

$$\begin{aligned}
\left(\frac{abc+bcd+cda+dab}{4}\right)^2 &\geq \left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)(abcd) \\
&\geq \sqrt{\left(\frac{a+b+c+d}{4}\right)\left(\frac{abc+bcd+cda+dab}{4}\right)}(abcd) \\
\Longrightarrow (abc+bcd+cda+dab)^3 &\geq 16a^2b^2c^2d^2(a+b+c+d) \qquad (**)
\end{aligned}$$

The desired is a direct consequence of (*) and (**).

47. (Vietnam 2002) Prove that for any reals x, y, z such that $x^2 + y^2 + z^2 = 9$,

$$2(x + y + z) - xyz \leq 10$$

Solution. Let $x^2 \geq y^2 \geq z^2$ so that $x^2 \geq 3$ and $6 \geq y^2 + z^2 \geq 2yz$. Cauchy's inequality gives

$$\begin{aligned}(2(x+y+z) - xyz)^2 &= (2(y+z) + x(2-yz))^2 \\ &\leq \left((y+z)^2 + x^2\right)\left(4 + (2-yz)^2\right) = (2yz+9)(y^2z^2 - 4yz + 8)\end{aligned}$$

Letting $\alpha = yz$, it would suffice to prove that $100 \geq (2\alpha+9)(\alpha^2 - 4\alpha + 8)$, but we have

$$100 - (2\alpha + 9)(\alpha^2 - 4\alpha + 8) = -2\alpha^3 - \alpha^2 + 20\alpha + 28 = (\alpha + 2)^2(7 - 2\alpha) \geq 0,$$

since $2\alpha \leq 6$, as desired. For equality we require $yz = -2$ and $x^2 = (2(y+z))^2 = 20 - 4x^2$ Thus, $x^2 = 4$ and $y + z = \pm 1$, which lead to $x = 2, y = 2, z = -1$.

48. (MOP 2003) For $n \geq 2$ a fixed positive integer, let $x_1, \ldots, x_n$ be positive reals such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1$$

Solution. We will prove the contrapositive. (We are motivated to do this for two good reasons: 1) it is usually difficult the show that the sum of some reciprocals is bounded above, and 2) the given relation in its current form is an abomination.) Take $y_i = \frac{1}{n-1+x_i}$, and for the sake of contradiction assume $y_1 + \cdots + y_n > 1$. Since the y_i are too large, the x_i are too small and we shall prove $\frac{1}{x_1} + \cdots + \frac{1}{x_n} > x_1 + \cdots + x_n$. Since $x_i y_i = 1 - (n-1)y_i$, we have

$$\begin{aligned}(n-1)y_i &> (n-1)\left(y_i + 1 - \sum_{j=1}^{n} y_j\right) \\ &= (n-1)y_i - 1 + \sum_{j=1}^{n}(1 - (n-1)y_j) \\ &= -x_i y_i + \sum_{j=1}^{n} x_j y_j \qquad (*) \\ \Longrightarrow \frac{n-1}{x_i} &> -1 + \sum_{j=1}^{n} \frac{x_j y_j}{x_i y_i} \qquad (**)\end{aligned}$$

Summing (**) over i,

$$(n-1)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right) > \sum_{i=1}^{n} x_i y_i \left(\left(\sum_{j=1}^{n} \frac{1}{x_j y_j}\right) - \frac{1}{x_i y_i}\right)$$

But by Cauchy and (*), we have

$$\left(\sum_{j=1}^{n} \frac{1}{x_j y_j}\right) - \frac{1}{x_i y_i} \geq \frac{(n-1)^2}{\left(\sum_{j=1}^{n} x_j y_j\right) - x_i y_i} > \frac{(n-1)^2}{(n-1)y_i} = \frac{n-1}{y_i}$$

Hence,

$$(n-1)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right) > \sum_{i=1}^{n} x_i y_i \left(\frac{n-1}{y_i}\right) = (n-1)(x_1+\cdots+x_n)$$

as desired.

49. (Taiwan 2002) Show that for all positive $a, b, c, d \leq k$, we have

$$\frac{a^4+b^4+c^4+d^4}{(2k-a)^4+(2k-b)^4+(2k-c)^4+(2k-d)^4} \geq \frac{abcd}{(2k-a)(2k-b)(2k-c)(2k-d)}$$

Solution. Without loss of generality, suppose that $a \geq b \geq c \geq d$. The desired is equivalent to

$$\begin{aligned} &\frac{(a^2-b^2)^2+(c^2-d^2)^2+2(a^2b^2+c^2d^2)}{abcd} \\ &\quad \geq \frac{1}{(2k-a)(2k-b)(2k-c)(2k-d)} \cdot \Big(\left((2k-a)^2-(2k-b)^2\right)^2 \\ &\qquad + \left((2k-c)^2+(2k-d)^2\right)^2 + 2\left((2k-a)^2(2k-b)^2+(2k-c)^2(2k-d)^2\right)\Big) \end{aligned}$$

We will realize this equivalent inequality as a sum of three simpler inequalities. We have

$$\begin{aligned} \frac{\left(a^2-b^2\right)^2}{abcd} &\geq \frac{\left((2k-a)^2-(2k-b)^2\right)^2}{(2k-a)(2k-b)(2k-c)(2k-d)} \qquad (*) \\ \Longleftarrow (a-b)^2\left[(a+b)^2(2k-a)(2k-b)\right] &\geq (a-b)^2\left[(4k-a-b)^2ab\right] \\ \Longleftarrow \left(\frac{a+b}{2}\right)^2(2k-a)(2k-b) &\geq \left(2k-\frac{a+b}{2}\right)^2 ab \\ \left(\frac{a+b}{2}\right)^2\left(\left(2k-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\right) &\geq \left(2k-\frac{a+b}{2}\right)^2\left(\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\right) \\ \left(\left(2k-\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2\right)\left(\frac{a-b}{2}\right)^2 &\geq 0 \end{aligned}$$

The two back-implications are consequences of $0 < c, d \leq k$ and division by $(a-b)^2$, respectively. As a consequence of $a, b \leq k$, it is evident that the last line holds, thus validating our inequality $(*)$. Substituting c and d for a and b shows that the related inequality

$$\frac{(c^2-d^2)^2}{abcd} \geq \frac{\left((2k-c)^2-(2k-d)^2\right)^2}{(2k-a)(2k-b)(2k-c)(2k-d)} \qquad (**)$$

holds as well.

We now verify our third inequality. Writing $f(x) = x+1/x$, we have that $f(x) = f(1/x)$ and that $f(x)$ is increasing for $x \geq 1$. Thus,

$$\begin{aligned}
\frac{2\left((2k-a)^2(2k-b)^2+(2k-c)^2(2k-d)^2\right)}{(2k-a)(2k-b)(2k-c)(2k-d)} &\leq \frac{2(a^2b^2+c^2d^2)}{abcd} \qquad (***)\\
\iff f\left(\frac{ab}{cd}\right) &\geq f\left(\frac{(2k-c)(2k-d)}{(2k-a)(2k-b)}\right)\\
\iff \frac{a(2k-a)}{c(2k-c)}\cdot\frac{b(2k-b)}{d(2k-d)} &\geq 1\\
\iff \left(\frac{k^2-(k-a)^2}{k^2-(k-c)^2}\right)\cdot\left(\frac{k^2-(k-b)^2}{k^2-(k-d)^2}\right) &\geq 1
\end{aligned}$$

It follows that $(***)$ holds as a consequence of $k \geq a \geq b \geq c \geq d > 0$, and that the desired inequality is realized by adding the corresponding sides of $(*)$, $(**)$, and $(***)$.

50. (Reid Barton, IMO Shortlist 03/A6.) Let $n \geq 2$ be a positive integer and $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ a sequence of $2n$ positive reals. Suppose $z_2, z_3, \ldots, z_{2n}$ is such that $z_{i+j}^2 \geq x_i y_j$ for all $i, j \in \{1, \ldots, n\}$. Let $M = \max\{z_2, z_3, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+z_3+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right)$$

Reid's official solution. Let $\max(x_1, \ldots, x_n) = \max(y_1, \ldots, y_n) = 1$. (We can do this by factoring X from every x_i, Y from every y_j, and $\sqrt{XY}$ from every z_{i+j} without changing the sign of the inequality.) We will prove $M + z_2 + \cdots + z_{2n} \geq x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n$, after which the desired follows by AM-GM. We will show that the number of terms on the left which are greater than r is at least as large as the number of terms on the right which are greater than r, for all $r \geq 0$.

For $r \geq 1$, the claim is obvious, since all terms on the right are at most 1. Now take $r < 1$. Let A and B denote the set of i for which $x_i > r$ and the set of j for which $y_j > r$ respectively, and write $a = |A|, b = |B|$. Evidently, from our scaling, $a, b \geq 1$. Now, $x_i > r$ and $y_j > r$ implies $z_{i+j} \geq \sqrt{x_i y_j} \geq r$. Hence, if C is the set of k for which $z_k > r$, we have $|C| \geq |A+B|$, where the set addition is defined by the set of possible values if we take an element of A and add it to an element of B. However, $|A+B| \geq |A|+|B|-1$, since if A and B consist of the values $p_1 < \cdots < p_a$ and

$q_1 < \cdots < q_b$ respectively we have all of the values $p_1+q_1 < \ldots < p_a+q_1 < \cdots < p_a+q_b$ in $A+B$. Hence, $|C| \geq a+b-1$. Since $|C| \geq 1$, there is some $z_k > r$, and hence, $M > r$. Therefore, the left side of the inequality in question has at least $a+b$ terms which exceed r, as desired. ■

The most difficult part of this problem is figuring out what can be done with the large number of givens, especially the mysterious M. Scaling allowed us to introduce some degree of control and, with audacity, a profoundly clever idea. As it turned out, the inequality was no sharper than simple AM-GM! It is my opinion that it is highly unlikely that a problem as staggeringly pernicious as this one will appear on an Olympiad - at least in the foreseeable future. Nevertheless, I have included it here for the purpose of illustrating just how unusual and creative a solution can be.

3 Problems

1. Show that for all positive reals a, b, c,

$$3a^2 + 3b^2 + 3c^2 \geq (a+b+c)^2$$

2. (MOP 01?) Show that for positive reals a, b, c,

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$

3. (MOP 04) Show that for all positive reals a, b, c,

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \geq 3$$

4. (MOP) Show that if k is a positive integer and $a_1, a_2, \ldots, a_n$ are positive reals which sum to 1, then

$$\prod_{i=1}^{n} \frac{1-a_i^k}{a_i^k} \geq \left(n^k - 1\right)^n$$

5. Let $a_1, a_2, \ldots, a_n$ be nonnegative reals with a sum of 1. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \leq \frac{1}{4}$$

6. (Ukraine 01) Let a, b, c, x, y, z be nonnegative reals such that $x+y+z=1$. Show that

$$ax + by + cz + 2\sqrt{(ab+bc+ca)(xy+yz+zx)} \leq a+b+c$$

7. Let $n > 1$ be a positive integer and $a_1, a_2, \ldots, a_n$ positive reals such that $a_1 a_2 \ldots a_n = 1$. Show that
$$\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n} \le \frac{a_1 + \cdots + a_n + n}{4}$$

8. (Aaron Pixton) Let a, b, c be positive reals with product 1. Show that
$$5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge (1+a)(1+b)(1+c)$$

9. (Valentin Vornicu[13]) Let a, b, c, x, y, z be arbitrary reals such that $a \ge b \ge c$ and either $x \ge y \ge z$ or $x \le y \le z$. Let $f : \mathbb{R} \to \mathbb{R}_0^+$ be either monotonic or convex, and let k be a positive integer. Prove that
$$f(x)(a-b)^k(a-c)^k + f(y)(b-c)^k(b-a)^k + f(z)(c-a)^k(c-b)^k \ge 0$$

10. (IMO 01/2) Let a, b, c be positive reals. Prove that
$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \ge 1$$

11. (Vasile Cirtoaje) Show that for positive reals a, b, c,
$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} + \frac{b^3}{(2b^2+c^2)(2b^2+a^2)} + \frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \le \frac{1}{a+b+c}$$

12. (USAMO 04/5) Let a, b, c be positive reals. Prove that
$$\left(a^5 - a^2 + 3\right)\left(b^5 - b^2 + 3\right)\left(c^5 - c^2 + 3\right) \ge (a+b+c)^3$$

13. (Titu Andreescu) Show that for all nonzero reals a, b, c,
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \ge \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

14. (Darij Grinberg) Show that for positive reals a, b, c,
$$\frac{b^2+c^2-a^2}{a(b+c)} + \frac{c^2+a^2-b^2}{b(c+a)} + \frac{a^2+b^2-c^2}{c(a+b)} \ge \frac{3}{2}$$

15. (IMO 96 Shortlist) Let a, b, c be positive reals with $abc = 1$. Show that
$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le 1$$

[13] This improvement is more widely known than the other one in this packet, and is published in his book, *Olimpiada de Matematica... de la provocare la experienta*, GIL Publishing House, Zalau, Romania. (In English, "The Math Olympiad... from challenge to experience.")

16. Let a, b, c be positive reals such that $a+b+c=1$. Prove that

$$\sqrt{ab+c}+\sqrt{bc+a}+\sqrt{ca+b} \geq 1+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$

17. (IMO 00/2) Positive reals a, b, c have product 1. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$$

18. (APMO 2005/2) Let a, b, c be positive reals with $abc=8$. Prove that

$$\frac{a^2}{\sqrt{(a^3+1)(b^3+1)}}+\frac{b^2}{\sqrt{(b^3+1)(c^3+1)}}+\frac{c^2}{\sqrt{(c^3+1)(a^3+1)}} \geq \frac{4}{3}$$

19. Show that for all positive reals a, b, c,

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c$$

20. (USAMO 97/5) Prove that for all positive reals a, b, c,

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

21. (Moldova 1999) Show that for all positive reals a, b, c,

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \geq \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}$$

22. (Tuymaada 2000) Prove that for all reals $0<x_1, \ldots, x_n \leq \frac{1}{2}$,

$$\left(\frac{n}{x_1+\cdots+x_n}-1\right)^n \leq \prod_{i=1}^n\left(\frac{1}{x_i}-1\right)$$

23. (Mathlinks Lore) Show that for all positive reals a, b, c, d with $abcd=1$, and $k \geq 2$,

$$\frac{1}{(1+a)^k}+\frac{1}{(1+b)^k}+\frac{1}{(1+c)^k}+\frac{1}{(1+d)^k} \geq 2^{2-k}$$

24. (Tiks) Show that for all reals $a, b, c>0$,

$$\frac{a^2}{2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}$$

25. (Hyun Soo Kim) Let a, b, c be positive reals with product not less than one. Prove that

$$\frac{1}{a+b^{2005}+c^{2005}}+\frac{1}{b+c^{2005}+a^{2005}}+\frac{1}{c+a^{2005}+b^{2005}} \leq 1$$

26. (IMO 05/3) Prove that for all positive a, b, c with product at least 1,

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{b^5+c^2+a^2}+\frac{c^5-c^2}{c^5+a^2+b^2} \geq 0$$

27. (Mildorf) Let a, b, c, k be positive reals. Determine a simple, necessary and sufficient condition for the following inequality to hold:

$$(a+b+c)^k\left(a^k b^k+b^k c^k+c^k a^k\right) \leq (ab+bc+ca)^k(a^k+b^k+c^k)$$

28. Let a, b, c be reals with $a+b+c=1$ and $a, b, c \geq -\frac{3}{4}$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{9}{10}$$

29. (Mildorf) Show that for all positive reals a, b, c,

$$\sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3} \leq \frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}$$

30. Let a, b, c, x, y, z be real numbers such that

$$(a+b+c)(x+y+z)=3, \quad (a^2+b^2+c^2)(x^2+y^2+z^2)=4$$

Prove that

$$ax+by+cz \geq 0$$

31. (Po-Ru Loh) Let a, b, c be reals with $a, b, c > 1$ such that

$$\frac{1}{a^2-1}+\frac{1}{b^2-1}+\frac{1}{c^2-1}=1$$

Prove that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \leq 1$$

32. (Weighao Wu) Prove that

$$(\sin x)^{\sin x} < (\cos x)^{\cos x}$$

for all real numbers $0 < x < \frac{\pi}{4}$.

33. (Michael Rozenberg) Show that for all positive reals a, b, c,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2} \cdot \frac{a^3+b^3+c^3}{a^2+b^2+c^2}$$

34. (Hungktn) Prove that for all positive reals a, b, c,

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

35. (Mock IMO 05/2) Let a, b, c be positive reals. Show that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leq \frac{3\sqrt{2}}{2}$$

36. (Gabriel Dospinescu) Let $n \geq 2$ be a positive integer. Show that for all positive reals $a_1, a_2, \ldots, a_n$ with $a_1 a_2 \ldots a_n = 1$,

$$\sqrt{\frac{a_1^2+1}{2}} + \cdots + \sqrt{\frac{a_n^2+1}{2}} \leq a_1 + \cdots + a_n$$

37. Let $n \geq 2$ be a positive integer, and let $k \geq \frac{n-1}{n}$ be a real number. Show that for all positive reals $a_1, a_2, \ldots, a_n$,

$$\left(\frac{(n-1)a_1}{a_2+\cdots+a_n}\right)^k + \left(\frac{(n-1)a_2}{a_3+\cdots+a_n+a_1}\right)^k + \cdots + \left(\frac{(n-1)a_n}{a_1+\cdots+a_{n-1}}\right)^k \geq n$$

38. Show that for reals x, y, z which are not all positive,

$$\frac{16}{9}\left(x^2-x+1\right)\left(y^2-y+1\right)\left(z^2-z+1\right) \geq (xyz)^2 - xyz + 1$$

39. (Mildorf) Let a, b, c be arbitrary reals such that $a \geq b \geq c$, and let x, y, z be nonnegative reals with $x + z \geq y$. Prove that

$$x^2(a-b)(a-c) + y^2(b-c)(b-a) + z^2(c-a)(c-b) \geq 0$$

and determine where equality holds.

40. (IMO 06/3) Determine the least real number M such that for all reals a, b, c,

$$\left|a^3b + b^3c + c^3a - a^3c - b^3a - c^3b\right| \leq M \cdot \left(a^2+b^2+c^2\right)^2$$

41. (Kiran Kedlaya) Show that for all nonnegative $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + \sqrt{a_1a_2} + \cdots + \sqrt[n]{a_1 \cdots a_n}}{n} \leq \sqrt[n]{a_1 \cdot \frac{a_1+a_2}{2} \cdots \frac{a_1+\cdots+a_n}{n}}$$

42. (Vasile Cirtoaje) Prove that for all positive reals a, b, c such that $a + b + c = 3$,

$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \geq \frac{3}{2}$$

43. (Gabriel Dospinescu) Prove that $\forall a, b, c, x, y, z \in \mathbb{R}^+ |\, xy + yz + zx = 3$,

$$\frac{a(y+z)}{b+c} + \frac{b(z+x)}{c+a} + \frac{c(x+y)}{a+b} \geq 3$$

44. (Mildorf) Let a, b, c be non-negative reals. Show that for all real k,

$$\sum_{cyc} \frac{\max(a^k, b^k)(a-b)^2}{2} \geq \sum_{cyc} a^k(a-b)(a-c) \geq \sum_{cyc} \frac{\min(a^k, b^k)(a-b)^2}{2}$$

(where $a, b, c \neq 0$ if $k \leq 0$) and determine where equality holds for $k > 0$, $k = 0$, and $k < 0$ respectively.

45. (Vasile Cirtoaje) Let a, b, c, k be positive reals. Prove that

$$\frac{ab+(k-3)bc+ca}{(b-c)^2+kbc} + \frac{bc+(k-3)ca+ab}{(c-a)^2+kca} + \frac{ca+(k-3)ab+bc}{(a-b)^2+kab} \geq \frac{3(k-1)}{k}$$

46. (Darij Grinberg and Vascile Cirtoaje) Show that for positive reals a, b, c, d,

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \geq \frac{2}{\sqrt{abcd}}$$

47. (Vasile Cirtoaje; inspired by the next problem) Show the for all positive reals a, b, c,

$$\frac{3a^2+ab}{(a+b)^2} + \frac{3b^2+bc}{(b+c)^2} + \frac{3c^2+ca}{(c+a)^2} \geq 3$$

48. (Vasile Cirtoaje; inspired by the next problem) Show that for all positive reals a, b, c,

$$\frac{3a^2-2ab-b^2}{a^2+b^2} + \frac{3b^2-2bc-c^2}{b^2+c^2} + \frac{3c^2-2ca-a^2}{c^2+a^2} \geq 0$$

49. (Mildorf) Show that for all positive reals a, b, c,

$$\frac{3a^2-2ab-b^2}{3a^2+2ab+3b^2} + \frac{3b^2-2bc-c^2}{3b^2+2bc+3c^2} + \frac{3c^2-2ca-a^2}{3c^2+2ca+3a^2} \geq 0$$

50. (Vasile Cirtoaje) Show that for real numbers a, b, c,

$$4\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right)\left(\sum_{cyc} a^4 - \sum_{cyc} a^2b^2\right) \geq 3\left(\sum_{cyc} a^3b - abc\sum_{cyc} a\right)^2$$